Announcements Feb 17

- Midterm 2 on March 6
- WeBWorK 2.7+2.9, 3.1 due Thursday
- My office hours Monday 3-4 and Wed 2-3
- Pop-up office hours Wed 3:30 – 4
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel ~~(different this week)~~
- Supplemental problems and practice exams on the master web site

- Come get your exam if you don't have it
- Will do midsemester feedback this week.

Sections 3.1

Matrix Transformations

Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$

From matrices to functions

Let A be an $m \times n$ matrix.

We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$T(v) = Av$$

This is called a matrix transformation.

The domain of T is $\mathbb{R}^n$.

The co-domain of T is $\mathbb{R}^m$.

The range of T is the set of outputs: $\mathrm{Col}(A)$

This gives us *another* point of view of $Ax = b$

A function

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

corresponds to

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

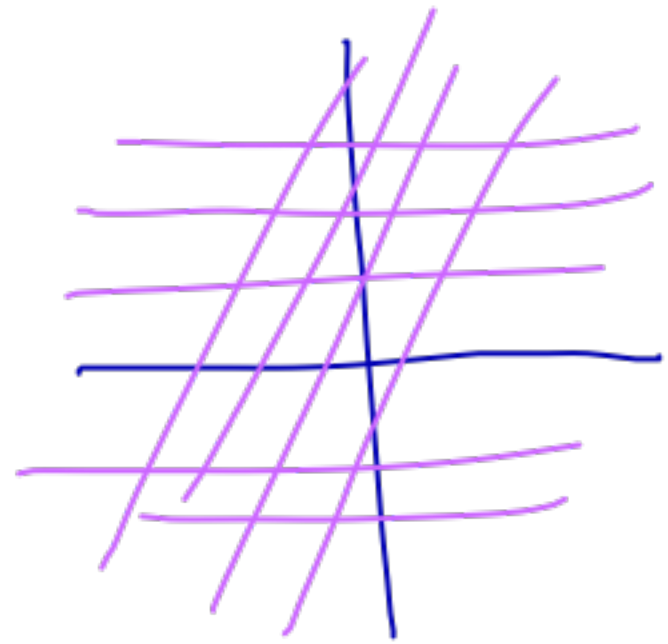
Demo

Square matrices

What does each matrix do to $\mathbb{R}^2$?

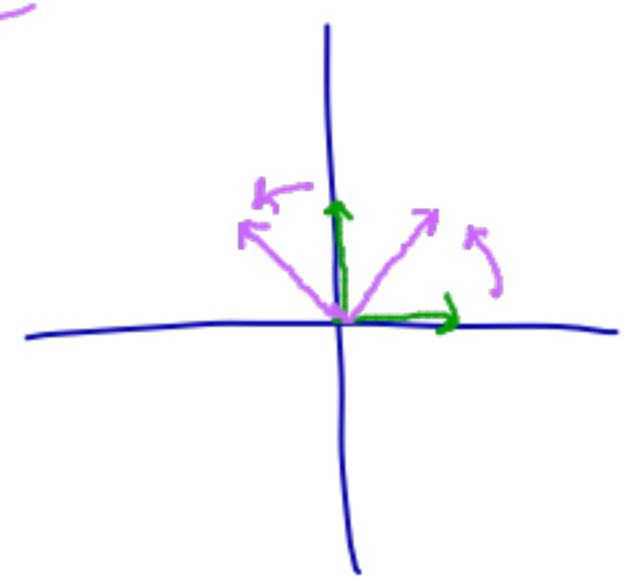
Hint: if you can't see it all at once, see what happens to the x- and y-axes.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ shear



$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

first col



rotate by $\pi/4$ and scale by $\sqrt{2}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ rotates by θ

Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

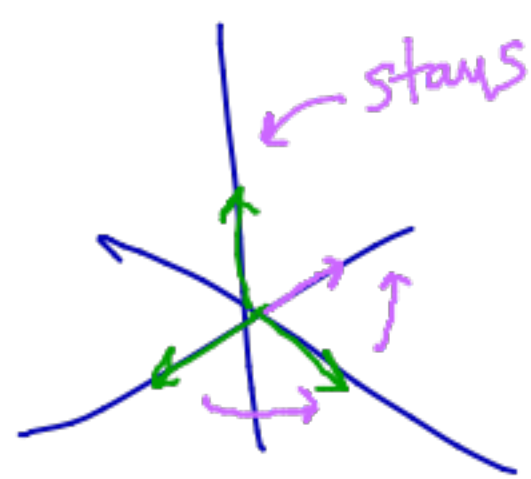
projection to xy-plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$$

refl. about xz-plane

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

rotate about z-axis by $\pi/2$



Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

Sections 3.2

One-to-one and onto transformations

Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ fn

One to one: for each output, at most one input.

From Calc: $f(x) = x$ ✓

$g(x) = x^2$ not one to one
$-3, 3$ have same output

Onto: at least one input for each element of co-domain

or: range = co-domain

From Calc
$f(x) = x$ $f : \mathbb{R} \to \mathbb{R}$ onto
$g(x) = x^2$ $g : \mathbb{R} \to \mathbb{R}$ not onto -7 not an output

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

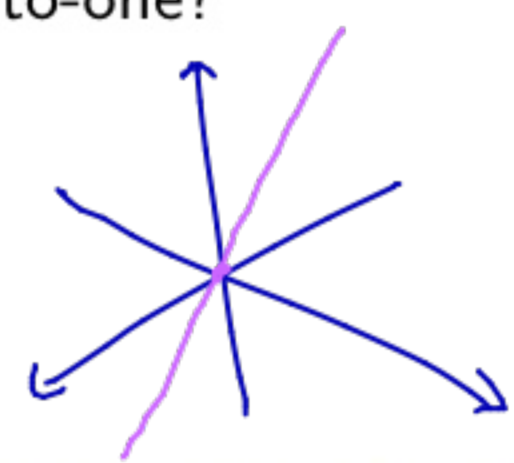
- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution
- A has a pivot in each column
- the range of T has dimension n

What can we say about the relative sizes of m and n if T is one-to-one?

tall: $m \geq n$

Draw a picture of the range of a one-to-one ~~mapping~~ matrix trans $\mathbb{R} \to \mathbb{R}^3$.

$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$



Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is onto
- the columns of A span $\mathbb{R}^m$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

What can we say about the relative sizes of m and n if T is onto?

wide: $m \leq n$

Give an example of an onto ~~mapping~~ matrix transf. $\mathbb{R}^3 \to \mathbb{R}$. $A = (1 \ \ 2 \ \ 3)$

One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

2 pivots | 2 pivots

$$\begin{pmatrix} \boxed{1} & 0 & 7 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{9} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

one-to-one	✓	✓	✗	✓
onto	✓	✗	✓	✓

Spot check T is onto for 3rd matrix:

For output $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ use input $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}$

Actually: any $\begin{pmatrix} 5 \\ 7 \\ z \end{pmatrix}$ works, so not one-to-one.

Example: $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

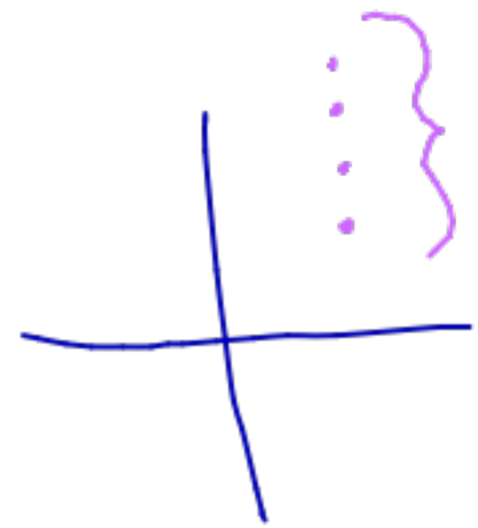
Not one-to-one or onto

One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$ projection



$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$ scaling

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ shear

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ rotation

Which are one to one / onto?

Poll

Which are one to one / onto?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

onto   one-to-one   neither

Demo

Demo

Demo

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are linearly independent
  - $Ax = 0$ has only the trivial solution
  - A has a pivot in each column
  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in each row
  - $Ax = b$ is consistent for all b in $\mathbb{R}^m$.
  - the range of T has dimension m

Section 3.3

Linear Transformations

Section 3.3 Outline

- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Find the matrix for a linear transformation

Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

Notice that $T(0) = 0$. *Why?*

We have the standard basis vectors for $\mathbb{R}^n$:

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \end{aligned}$$

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. *Why?*

In engineering, this is called the principle of superposition.

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix A so that

$$T(v) = Av$$

for all v in $\mathbb{R}^n$.

The matrix for a linear transformation is called the standard matrix.

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

Why? Notice that $Ae_i = T(e_i)$ for all i. Then it follows from linearity that $T(v) = Av$ for all v.

The identity

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$T(v) = v$$

What is the standard matrix?

This standard matrix is called I_n or I.

Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

What is the standard matrix for T?

In fact, a function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables).

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the x-direction and 3 in the y-direction, and then reflects over the line $y = x$.

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.

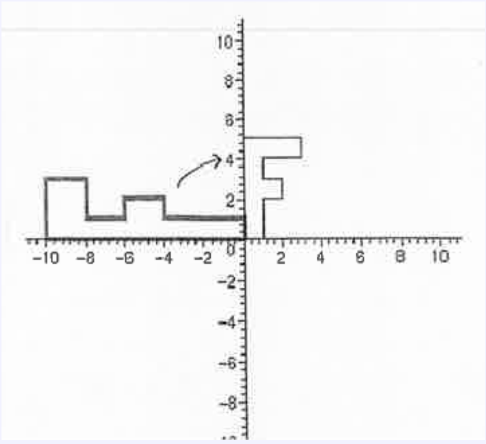
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion

Discussion Question

Find a matrix that does this.



Summary of 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.