Announcements Feb 17

- Midterm 2 on March 6
- WeBWorK 2.7+2.9, 3.1 due Thursday
- My office hours Monday 3-4 and Wed 2-3
- Pop-up office hours Wed 3:30 – 4
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45

- PLUS sessions Mon/Wed 6-7 LLC West with Miguel (different this week)
- Supplemental problems and practice exams on the master web site

  - Come get your exam if you don't have it
  - Will do midterm feedback this week.
Sections 3.1
Matrix Transformations
Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$
From matrices to functions

Let $A$ be an $m \times n$ matrix.

We define a function

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$T(v) = Av$$

This is called a matrix transformation.

The **domain** of $T$ is $\mathbb{R}^n$.

The **co-domain** of $T$ is $\mathbb{R}^m$.

The **range** of $T$ is the set of outputs: $\text{Col}(A)$

This gives us another point of view of $Ax = b$
Square matrices

What does each matrix do to $\mathbb{R}^2$?

Hint: if you can’t see it all at once, see what happens to the $x$- and $y$-axes.

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\text{ shear}
\]

\[
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\text{ first col}
\]

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\text{ rotates by } \theta
\]

rotate by $\pi/4$ and scale by $\sqrt{2}$
Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
x \\
y \\
0
\end{pmatrix}
\]
projection to $xy$-plane

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
x \\
y \\
-z
\end{pmatrix}
\]
refl. about $xz$-plane

\[
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
-y \\
x \\
z
\end{pmatrix}
\]
rotate about $z$-axis by $\pi/2$

rotate about $z$-axis by $\pi/2$

rotate about $z$-axis by $\pi/2$

rotate about $z$-axis by $\pi/2$

rotate about $z$-axis by $\pi/2$

rotate about $z$-axis by $\pi/2$

rotate about $z$-axis by $\pi/2$

rotate about $z$-axis by $\pi/2$

rotate about $z$-axis by $\pi/2$
Section 3.1 Summary

- If $A$ is an $m \times n$ matrix, then the associated matrix transformation $T$ is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\text{Col}(A)$.

- If $A$ is $n \times n$ then $T$ does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation.
Sections 3.2

One-to-one and onto transformations
Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ fn} \]

One to one: for each output, at most one input.

From Calc:
- \( f(x) = x \) \( \checkmark \)
- \( g(x) = x^2 \) not one to one
  \(-3, 3\) have same output

Onto: at least one input for each element of co-domain
  or: range = co-domain

From Calc:
- \( f(x) = x \) \( f : \mathbb{R} \rightarrow \mathbb{R} \) onto
- \( g(x) = x^2 \) \( g : \mathbb{R} \rightarrow \mathbb{R} \) not onto
  \(-7 \) not an output
One-to-one

\( T : \mathbb{R}^n \to \mathbb{R}^m \) is one-to-one if each \( b \) in \( \mathbb{R}^m \) is the output for at most one \( v \) in \( \mathbb{R}^n \).

In other words: different inputs have different outputs.

**Theorem.** Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:

- \( T \) is one-to-one
- the columns of \( A \) are linearly independent
- \( Ax = 0 \) has only the trivial solution
- \( A \) has a pivot in each column
- the range of \( T \) has dimension \( n \)

What can we say about the relative sizes of \( m \) and \( n \) if \( T \) is one-to-one?

\[ \forall \text{all: } m \geq n \]

Draw a picture of the range of a one-to-one mapping \( \mathbb{R} \to \mathbb{R}^3 \).

\[ A = \left( \begin{array}{c} \frac{1}{2} \\ 0 \\ \frac{1}{3} \end{array} \right) \]
Onto

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto if the range of $T$ equals the codomain $\mathbb{R}^m$, that is, each $b$ in $\mathbb{R}^m$ is the output for at least one input $v$ in $\mathbb{R}^m$.

**Theorem.** Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation with matrix $A$. Then the following are all equivalent:

- $T$ is onto
- the columns of $A$ span $\mathbb{R}^m$
- $A$ has a pivot in each row
- $Ax = b$ is consistent for all $b$ in $\mathbb{R}^m$
- the range of $T$ has dimension $m$

What can we say about the relative sizes of $m$ and $n$ if $T$ is onto?

**wide:** $m \leq n$

Give an example of an onto mapping $\mathbb{R}^3 \rightarrow \mathbb{R}$. $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$
One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

\[
\begin{pmatrix}
1 & 0 & 7 \\
0 & 1 & 2 \\
0 & 0 & 9
\end{pmatrix} \quad \text{2 pivots} \quad \begin{pmatrix}
1 & 0 \\
1 & 1 \\
2 & 1
\end{pmatrix} \quad \text{2 pivots} \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1
\end{pmatrix} \quad \text{2 pivots}
\]

- One-to-one: ✔ ✔ ✗ ✔
- Onto: ✔ ✗ ✔ ✔

Spot check $T$ is onto for 3rd matrix:

For output \( \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix} \) use input \( \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \)

Actually, any \( \begin{pmatrix} 5 \\ 1 \\ z \end{pmatrix} \) works, so not one-to-one.

Example: $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 \end{pmatrix}$

Not one-to-one or onto
One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} \quad \text{reflection}
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
\end{pmatrix} \quad \text{projection}
\]

\[
\begin{pmatrix}
3 & 0 \\
0 & 3 \\
\end{pmatrix} \quad \text{scaling}
\]

\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix} \quad \text{shear}
\]

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{pmatrix} \quad \text{rotation}
\]
Which are one to one / onto?

- \( \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \)
- \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \)
- \( \begin{pmatrix} 1 & -2 & -1 & 2 \\ -1 & 2 & -4 \end{pmatrix} \)

- onto
- one-to-one
- neither
Summary of Section 3.2

- $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if each $b$ in $\mathbb{R}^m$ is the output for at most one $v$ in $\mathbb{R}^n$.

- **Theorem.** Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation with matrix $A$. Then the following are all equivalent:
  - $T$ is one-to-one
  - the columns of $A$ are linearly independent
  - $Ax = 0$ has only the trivial solution
  - $A$ has a pivot in each column
  - the range has dimension $n$

- $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** if the range of $T$ equals the codomain $\mathbb{R}^m$, that is, each $b$ in $\mathbb{R}^m$ is the output for at least one input $v$ in $\mathbb{R}^n$.

- **Theorem.** Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a matrix transformation with matrix $A$. Then the following are all equivalent:
  - $T$ is onto
  - the columns of $A$ span $\mathbb{R}^m$
  - $A$ has a pivot in each row
  - $Ax = b$ is consistent for all $b$ in $\mathbb{R}^m$.
  - the range of $T$ has dimension $m$
Section 3.3

Linear Transformations
Section 3.3 Outline

- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Find the matrix for a linear transformation
Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u + v) = T(u) + T(v)$ for all $u, v$ in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all $v$ in $\mathbb{R}^n$ and $c$ in $\mathbb{R}$.

Notice that $T(0) = 0$. Why?

We have the standard basis vectors for $\mathbb{R}^n$:

$$e_1 = (1, 0, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. Why?

In engineering, this is called the principle of superposition.
Linear transformations are matrix transformations

**Theorem.** Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix $A$ so that

$$T(v) = Av$$

for all $v$ in $\mathbb{R}^n$.

The matrix for a linear transformation is called the **standard matrix**.
Linear transformations are matrix transformations

**Theorem.** Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix}
T(e_1) & T(e_2) & \cdots & T(e_n)
\end{pmatrix}$$

Why? Notice that $Ae_i = T(e_i)$ for all $i$. Then it follows from linearity that $T(v) = Av$ for all $v$. 
The identity

The identity linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is

$$T(v) = v$$

What is the standard matrix?

This standard matrix is called $I_n$ or $I$. 
Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + y \\ y \\ x - y \end{array} \right)$$

What is the standard matrix for $T$?

In fact, a function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables).
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the $x$-direction and 3 in the $y$-direction, and then reflects over the line $y = x$. 
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the $y$-axis and then rotates counterclockwise by $\pi/2$. 
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane.
Discussion Question

Find a matrix that does this.
Summary of 3.3

- A function \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is linear if
  - \( T(u + v) = T(u) + T(v) \) for all \( u, v \) in \( \mathbb{R}^n \).
  - \( T(cv) = cT(v) \) for all \( v \in \mathbb{R}^n \) and \( c \) in \( \mathbb{R} \).

- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).

- The standard matrix for a linear transformation has its 1st column equal to \( T(e_i) \).