Announcements Feb 19

- Midterm 2 on March 6
- WeBWorK 2.7+2.9, 3.1 due Thursday
- My office hours Monday 3-4 and Wed 2-3 in Skiles 234
- Pop-up office hours Wed 3:30-4 this week in Skiles 234
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems and practice exams on the master web site

Why are we learning about matrix transformations?

Sample applications:

- Cryptography (Hill cypher)
- Computer graphics (Perspective projection is a linear map!)
- Aerospace (Control systems - input/output)
- Biology
- Many more!

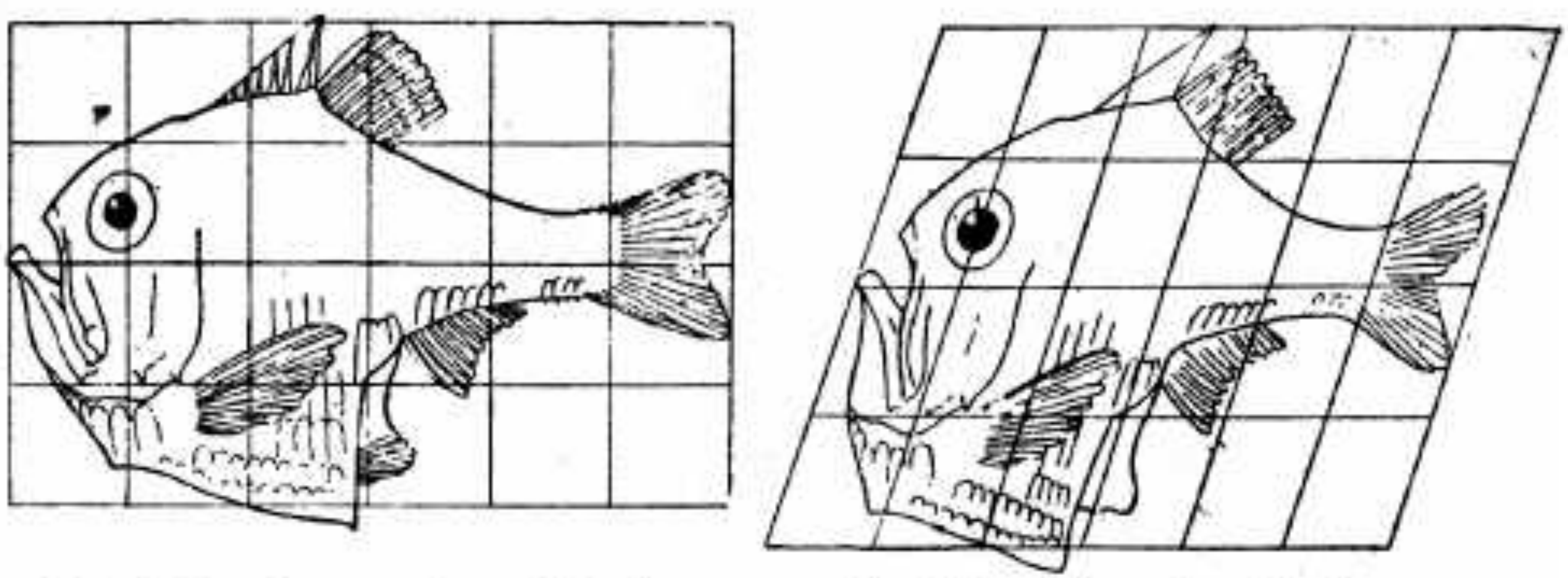

Fig. 517. *Argyropelecus Olfersi.* Fig. 518. *Sternoptyx diaphana.*

Section 3.3

Linear Transformations

Section 3.3 Outline

- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Find the matrix for a linear transformation

Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

First examples: matrix transformations.

A = m×n matrix

⇝ $T(v) = Av$

This is a lin transf because:

$T(u+v) = A(u+v) = Au + Av = T(u) + T(v)$

Same for 2nd bullet

Linear transformations

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Notice that $T(0) = 0$. *Why?*

$T(0) = T(0 \cdot v) = 0 \cdot T(v) = 0$

We have the standard basis vectors for $\mathbb{R}^n$:

$$e_1 = (1, 0, 0, \ldots, 0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$e_2 = (0, 1, 0, \ldots, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$\vdots$$

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. *Why?*

$T(5e_1 + 7e_2) = 5T(e_1) + 7T(e_2)$

In engineering, this is called the principle of superposition.

$f(x) = x^2$

$f(5)$

$f(x) = 2x$ ✓

$f(x) = 2x + 1$ ✗

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\rightsquigarrow T : \mathbb{R}^2 \to \mathbb{R}^2$

$v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$T(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix A so that

$$T(v) = Av$$

for all v in $\mathbb{R}^n$.

The matrix for a linear transformation is called the standard matrix.

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix}$$

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

$$Ae_1 = T(e_1) = 1^{st} \text{ col.}$$
$$Ae_2 = T(e_2) = 2^{nd} \text{ col.}$$

Why? Notice that $Ae_i = T(e_i)$ for all i. Then it follows from linearity that $T(v) = Av$ for all v.

The identity

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$T(v) = v$$

$$T(u+v) = u+v = T(u) + T(v)$$

What is the standard matrix?

This standard matrix is called I_n or I.

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

Check

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ w \end{pmatrix}\right) = T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} z \\ w \end{pmatrix}$$

$$T\begin{pmatrix} x+z \\ y+w \end{pmatrix} = \begin{pmatrix} x+z+y+w \\ y+w \\ x+z-(y+w) \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$$

What is the standard matrix for T?

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Check

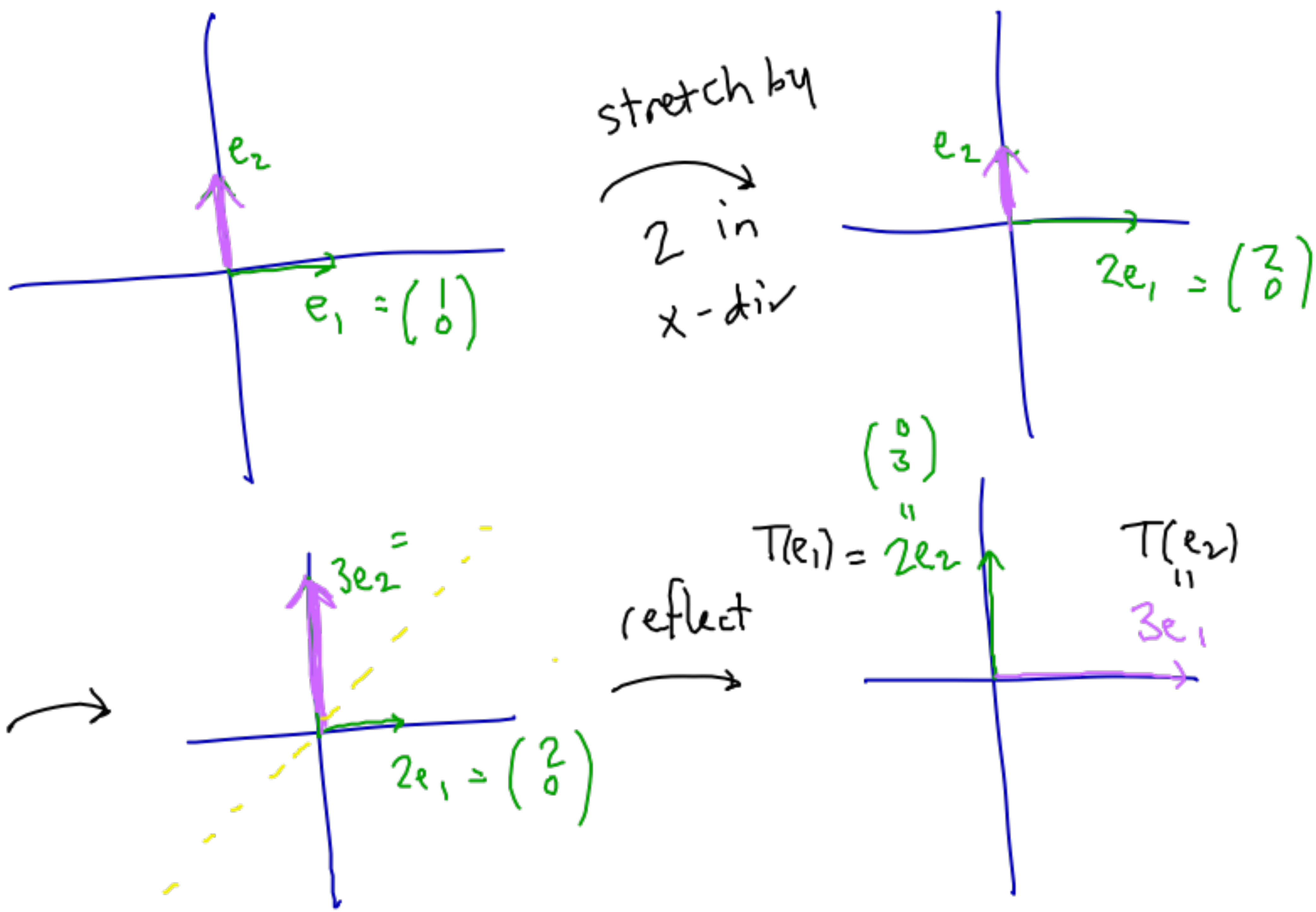
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

In fact, a function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables, no constant terms).

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+3y \\ x \end{pmatrix} \longleftrightarrow \begin{pmatrix} 5 & 3 \\ 1 & 0 \end{pmatrix}$$

Linear transformations are matrix transformations

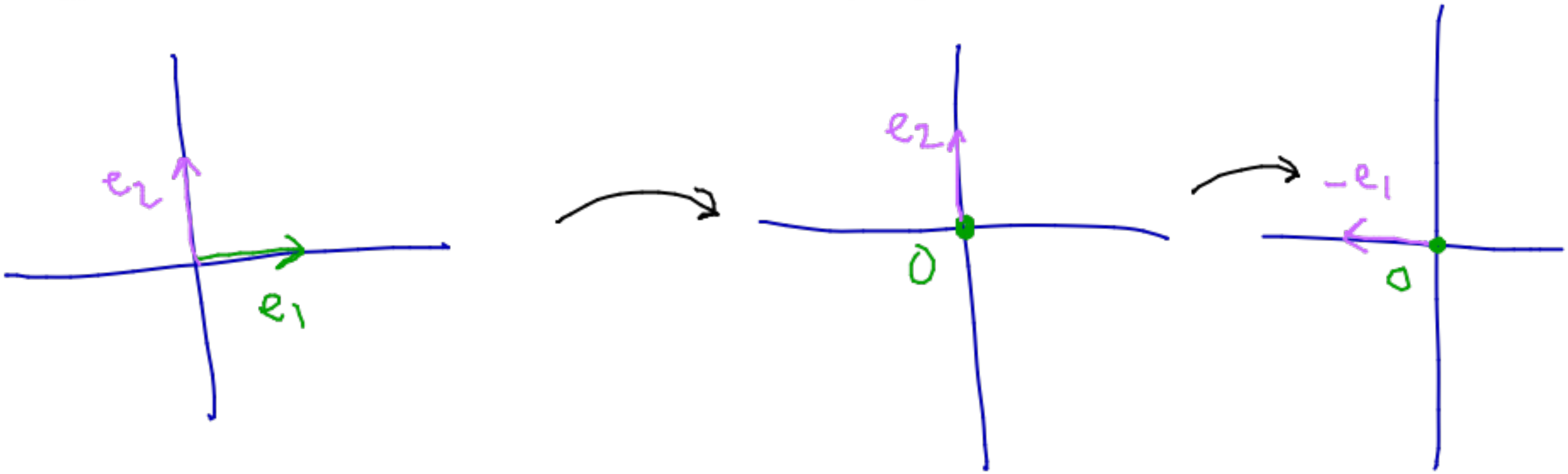
Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the x-direction and 3 in the y-direction, and then reflects over the line $y = x$.



$$\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$$

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.



$$\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

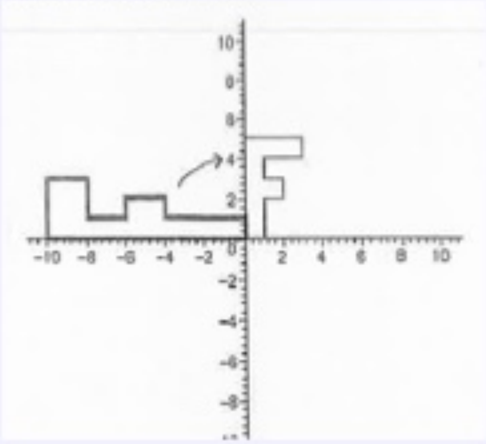
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion

Discussion Question

Find a matrix that does this.



$$\begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$$

Summary of 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.

Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things

Function composition

Remember from calculus that if f and g are functions then the composition $f \circ g$ is a new function defined as follows:

$$f \circ g(x) = f(g(x))$$

In words: first apply g, then f.

Example: $f(x) = x^2$ and $g(x) = x + 1$.

Note that $f \circ g$ is usually different from $g \circ f$.

Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^m \to \mathbb{R}^p$ and $U : \mathbb{R}^n \to \mathbb{R}^m$ and make the composition $T \circ U$.

Notice that both have an m. Why?

What are the domain and codomain for $T \circ U$?

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?

Composition of linear transformations

Example. $T =$ projection to y-axis and
$U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

What about $U \circ T$?

Matrix Multiplication

And now for something completely different (not really!)

Suppose A is an $m \times n$ matrix. We write a_{ij} or A_{ij} for the ijth entry.

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where r_i is the ith row of A, and b_j is the jth column of B.

Or: the jth column of AB is A times the jth column of B.

Multiply these matrices (both ways):

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix}$$

Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do U then do T

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case Bv is the ith column of B. So the left-hand side is A times the ith column of B. The right-hand side is the ith column of AB which we already said was A times the ith column of B. It works!

Matrix Multiplication and Linear Transformations

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

Example. $T =$ projection to y-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion Question

Are there nonzero matrices A and B with $AB = 0$?

1. Yes
2. No

Properties of Matrix Multiplication

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$
- $(AB)^T = B^T A^T$
- $I_m A = A = AI_n$, where I_k is the $k \times k$ identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- AB is not always equal to BA
- $AB = AC$ does not mean that $B = C$
- $AB = 0$ does not mean that A or B is 0

Sums and Scalar Multiples

Same as for vectors: component-wise, so matrices must be same size to add.

$A + B = B + A$

$(A + B) + C = A + (B + C)$

$r(A + B) = rA + rB$

$(r + s)A = rA + sA$

$(rs)A = r(sA)$

$A + 0 = A$

(We can define linear transformations $T + U$ ad cT, and so all of the above facts are also facts about linear transformations.)

Summary of Section 3.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do U then T)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the ith column of AB is $A(b_i)$
- Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.
- Warning!
  - AB is not always equal to BA
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that A or B is 0