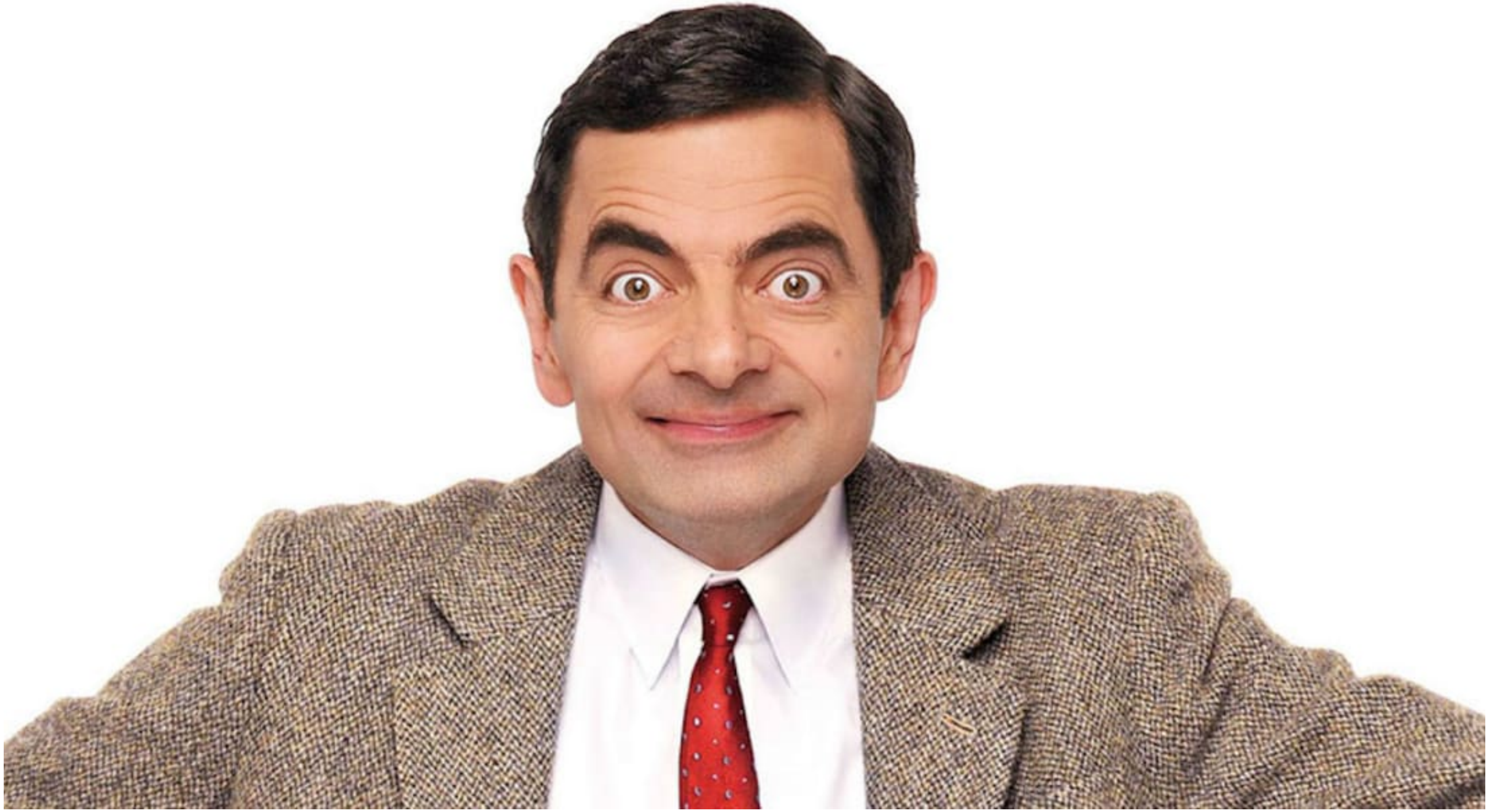
# Announcements Feb 19

- Midterm 2 on March 6 3.2 & 3.3
- WeBWorK ~~2.7–2.9, 3.1~~ due Thursday
- Mid-semester evaluation under Quizzes on Canvas (due today)
- My office hours Monday 3-4 and Wed 2-3 in Skiles 234
- Pop-up office hours Wed 11-11:30 this week in Skiles 234
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems and practice exams on the master web site
- Quiz on 3.2, 3.3 on Fri



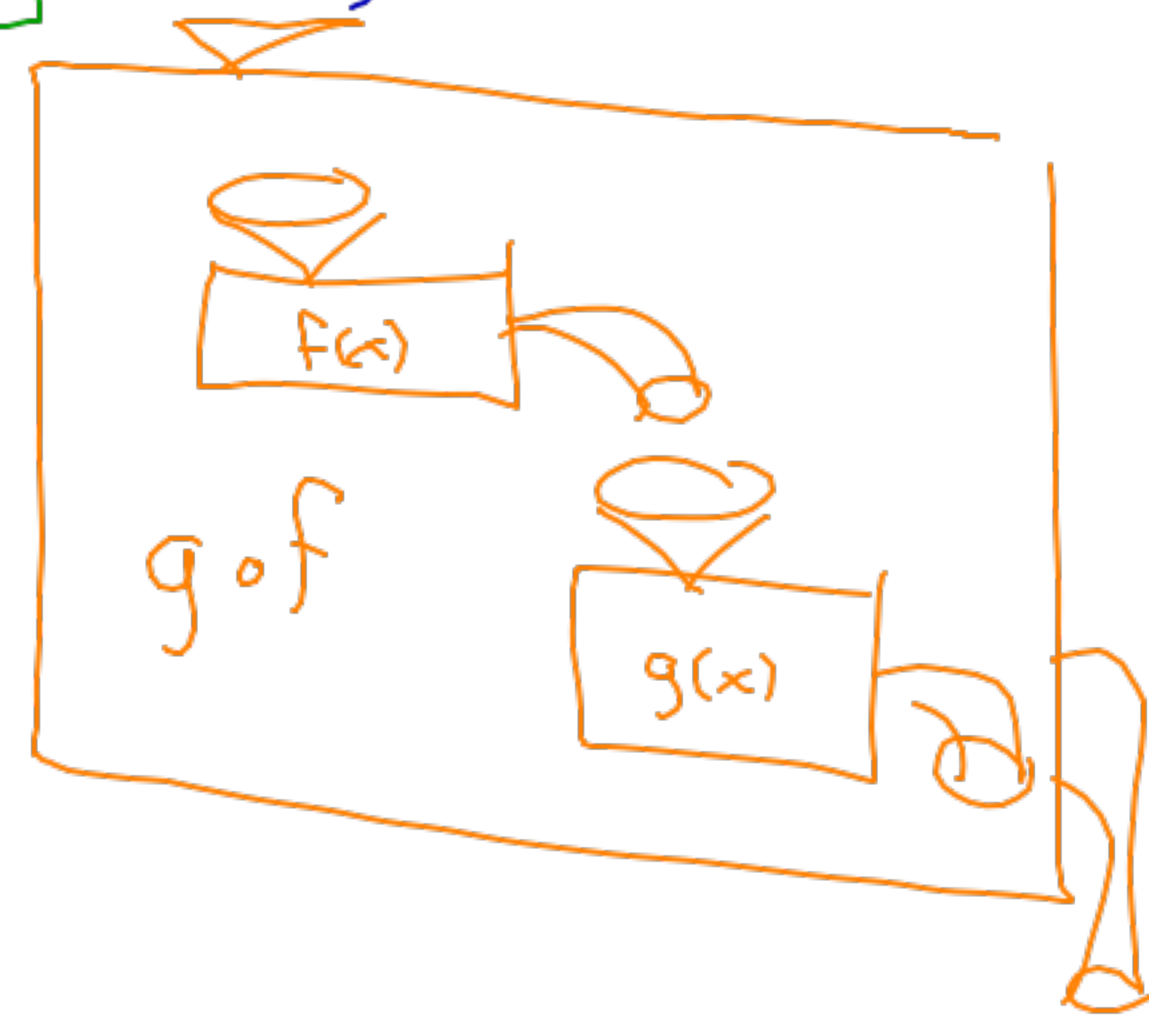
# Section 3.4

## Matrix Multiplication

# Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things

Composition: $\boxed{g \circ f}(x) = g(f(x))$




Example: $f(x) = x+1$

$g(x) = x^2$

$\longrightarrow$ $g \circ f(x) = (x+1)^2$

$f \circ g(x) = x^2 + 1$

# Function composition

Remember from calculus that if $f$ and $g$ are functions then the composition $f \circ g$ is a new function defined as follows:

$$f \circ g(x) = f(g(x))$$

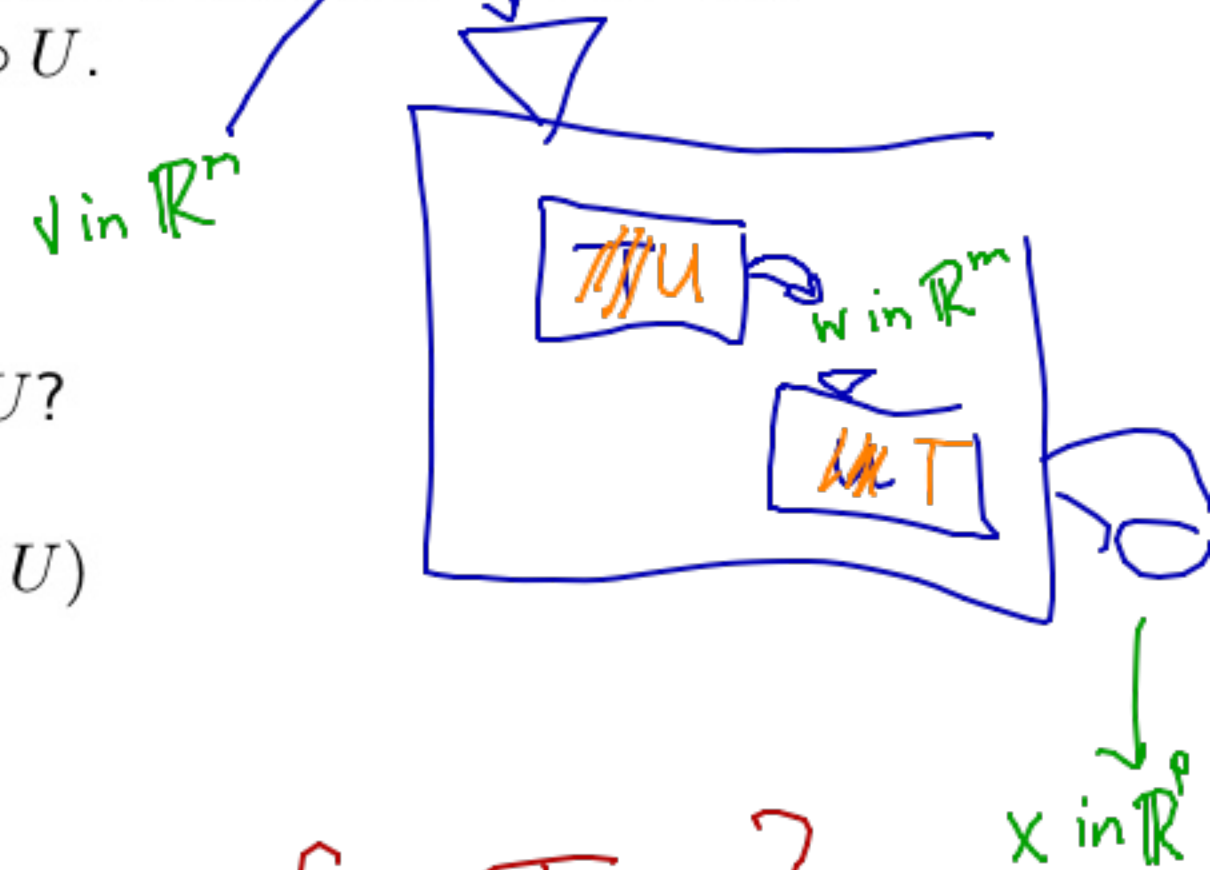
In words: first apply $g$, then $f$.

Example: $f(x) = x^2$ and $g(x) = x + 1$.

Note that $f \circ g$ is usually different from $g \circ f$.

## Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^m \to \mathbb{R}^p$ and $U : \mathbb{R}^n \to \mathbb{R}^m$ and make the composition $T \circ U$.




Notice that both have an $m$. Why?

What are the domain and codomain for $T \circ U$?

$\mathbb{R}^n$ $\mathbb{R}^p$

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

*Why?*

What is the matrix for $T \circ U$?

We know: $p \times n$

range contained in range of $T$

# Composition of linear transformations

**Example.** $T$ = projection to $y$-axis and
$U$ = reflection about $y = x$ in $\mathbb{R}^2$

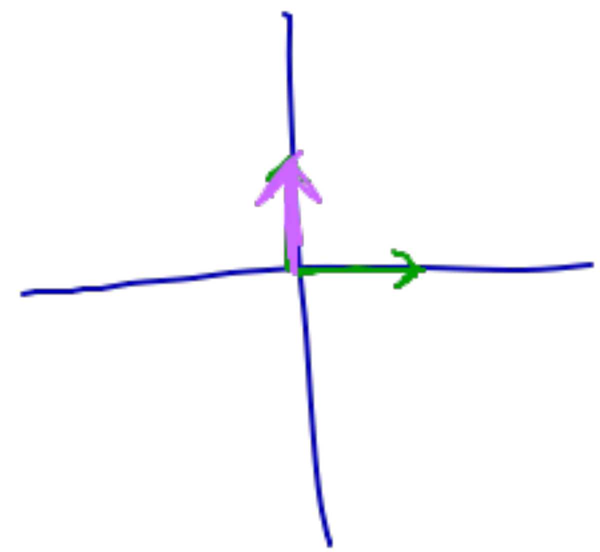
$T : \mathbb{R}^2 \to \mathbb{R}^2$
$U : \mathbb{R}^2 \to \mathbb{R}^2$

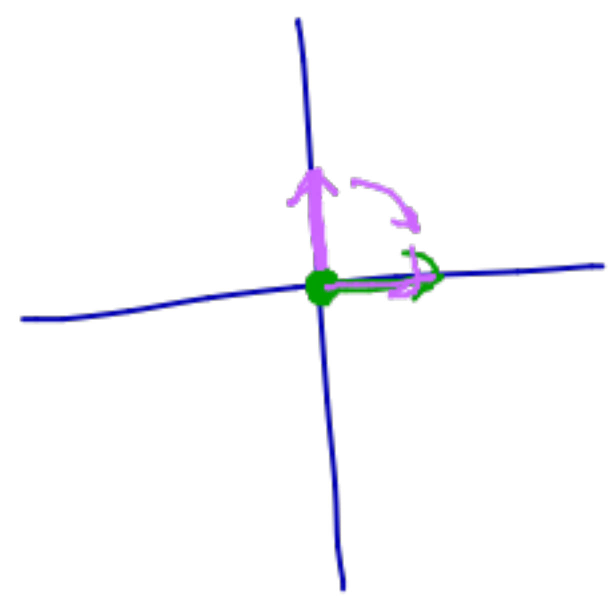
What is the standard matrix for $T \circ U$?

What about $U \circ T$?

$$T \circ U \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$U \circ T \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$





# Matrix Multiplication

## And now for something completely different (not really!)

Suppose $A$ is an $m \times n$ matrix. We write $a_{ij}$ or $A_{ij}$ for the $ij$th entry.

If $A$ is $m \times n$ and $B$ is $n \times p$, then $AB$ is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where $r_i$ is the $i$th row of $A$, and $b_j$ is the $j$th column of $B$.

Or: the $j$th column of $AB$ is $A$ times the $j$th column of $B$.

Multiply these matrices (both ways):

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}_{2 \times \cancel{3}} \underbrace{\begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix}}_{\cancel{3} \times 2} = \underbrace{\begin{pmatrix} 8 & -4 \\ 17 & -13 \end{pmatrix}}_{2 \times 2}$$

# Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do $U$ then do $T$

Fact. Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is $AB$.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case $Bv$ is the $i$th column of $B$. So the left-hand side is $A$ times the $i$th column of $B$. The right-hand side is the $i$th column of $AB$ which we already said was $A$ times the $i$th column of $B$. It works!

# Matrix Multiplication and Linear Transformations

**Fact.** Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is $AB$.

**Example.** $T =$ projection to $y$-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

Matrix for $T$ : $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Matrix for $U$ : $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

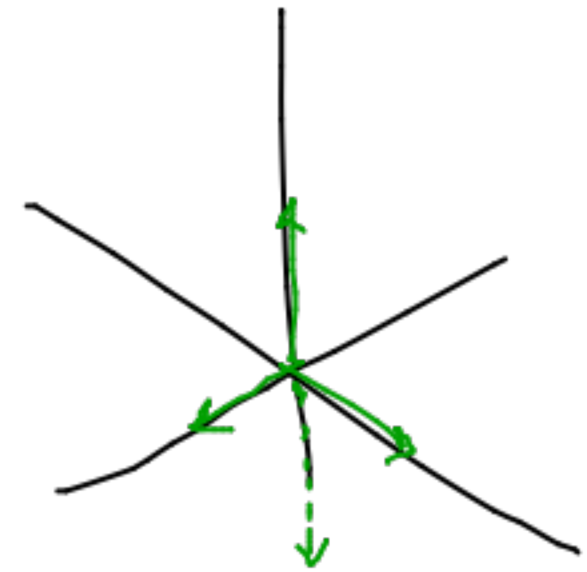
Matrix for $T \circ U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$U \circ T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

# Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane.

$T$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $U$



$$T \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$U \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$U \circ T \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$T \circ U \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

**Discussion Question**

Are there nonzero matrices $A$ and $B$ with $AB = 0$?

1. Yes
2. No

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

# Properties of Matrix Multiplication

- $A(BC) = (AB)C$ b/c composition of L.T.'s $\longleftrightarrow$ mult of mat's
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$ r in $\mathbb{R}$ (scalar)
- $\cancel{(AB)^T = B^T A^T}$
- $I_m A = A = A I_n$, where $I_k$ is the $k \times k$ identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- $AB$ is not always equal to $BA$
- $AB = AC$ does not mean that $B = C$
- $AB = 0$ does not mean that $A$ or $B$ is 0

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&0\end{pmatrix} = 0$$

## Sums and Scalar Multiples

Same as for vectors: component-wise, so matrices must be same size to add.

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$(rs)A = r(sA)$$

$$A + 0 = A$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(We can define linear transformations $T + U$ ad $cT$, and so all of the above facts are also facts about linear transformations.)

# Summary of Section 3.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do $U$ then $T$)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the $i$th column of $AB$ is $A(b_i)$
- Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is $AB$.
- Warning!
  - $AB$ is not always equal to $BA$
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that $A$ or $B$ is 0

# Section 3.5 Outline

- The definition of a matrix inverse
- How to find a matrix inverse
- Inverses for linear transformations

$$7x = 35$$

$$7^{-1} \cdot 7x = 7^{-1} \cdot 35$$

$$1 \cdot x = 5$$

# Inverses

To solve

$$Ax = b$$

we might want to "divide both sides by $A$".

We will make sense of this...

## Inverses

$A = n \times n$ matrix.

$A$ is invertible if there is a matrix $B$ with

$$AB = BA = I_n$$

$B$ is called the inverse of $A$ and is written $A^{-1}$

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## The $2 \times 2$ Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of $A$.

*Fact.* If $\det(A) \neq 0$ then $A$ is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then $A$ is not invertible.

$$\begin{pmatrix} 5 & 10 \\ 7 & 14 \end{pmatrix} = 5 \cdot 14 - 7 \cdot 10 = 0$$

not invertible!

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

$$-\frac{1}{2} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

## Solving Linear Systems via Inverses

Fact. If $A$ is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

Solve

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

# Solving Linear Systems via Inverses

What if we change $b$?

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 0 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

So finding the inverse is essentially the same as solving all $Ax = b$ equations at once (fixed $A$, varying $b$).