Announcements Feb 19

- Midterm 2 on **March 6**
- WeBWorK due Thursday
- Mid-semester evaluation under Quizzes on Canvas (due today)
- My office hours Monday 3-4 and Wed 2-3 in Skiles 234
- Pop-up office hours Wed 11-11:30 this week in Skiles 234
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems and practice exams on the master web site

- **Quiz on 3.2, 3.3 on Fri**
Section 3.4
Matrix Multiplication
Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things

**Composition:** \( g \circ f(x) = g(f(x)) \)

**Example:**
\[
\begin{align*}
  f(x) &= x + 1 \\
  g(x) &= x^2 \\
  \Rightarrow \quad (g \circ f)(x) &= (x+1)^2 \\
  f \circ g(x) &= x^2 + 1
\end{align*}
\]
Function composition

Remember from calculus that if \( f \) and \( g \) are functions then the composition \( f \circ g \) is a new function defined as follows:

\[
f \circ g(x) = f(g(x))
\]

In words: first apply \( g \), then \( f \).

Example: \( f(x) = x^2 \) and \( g(x) = x + 1 \).

Note that \( f \circ g \) is usually different from \( g \circ f \).
Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^m \to \mathbb{R}^p$ and $U : \mathbb{R}^n \to \mathbb{R}^m$ and make the composition $T \circ U$.

Notice that both have an $m$. Why?

What are the domain and codomain for $T \circ U$?

$\mathbb{R}^n \quad \mathbb{R}^p$

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?

What is the matrix for $T \circ U$?

We know: $p \times n$

range contained in range of $T$
Composition of linear transformations

Example. $T =$ projection to $y$-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

What about $U \circ T$?

$T \circ U \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$U \circ T \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
Matrix Multiplication
And now for something completely different (not really!)

Suppose $A$ is an $m \times n$ matrix. We write $a_{ij}$ or $A_{ij}$ for the $ij$th entry.

If $A$ is $m \times n$ and $B$ is $n \times p$, then $AB$ is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where $r_i$ is the $i$th row of $A$, and $b_j$ is the $j$th column of $B$.

Or: the $j$th column of $AB$ is $A$ times the $j$th column of $B$.

Multiply these matrices (both ways):

$$
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
0 & -2 \\
1 & -1 \\
2 & 0
\end{pmatrix}
= 
\begin{pmatrix}
8 & -4 \\
17 & -13
\end{pmatrix}
$$

$2 \times 3$ $3 \times 2$ $2 \times 2$
Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do $U$ then do $T$

**Fact.** Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is $AB$.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case $Bv$ is the $i$th column of $B$. So the left-hand side is $A$ times the $i$th column of $B$. The right-hand side is the $i$th column of $AB$ which we already said was $A$ times the $i$th column of $B$. It works!
Matrix Multiplication and Linear Transformations

Fact. Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is $AB$.

Example. $T = \text{projection to } y\text{-axis}$ and $U = \text{reflection about } y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

Matrix for $T : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Matrix for $U : \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$T \circ U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$U \circ T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane.

\[ T \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ U \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ U \circ T \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ T \circ U \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Discussion Question

Are there nonzero matrices $A$ and $B$ with $AB = 0$?

1. Yes
2. No

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]
Properties of Matrix Multiplication

- \( A(BC) = (AB)C \)
- \( A(B + C) = AB + AC \)
- \( (B + C)A = BA + CA \)
- \( r(AB) = (rA)B = A(rB) \)
- \( (AB)^T = BTAT \)
- \( I_mA = A = AI_n \), where \( I_k \) is the \( k \times k \) identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

**Warning!**

- \( AB \) is not always equal to \( BA \)
- \( AB = AC \) does not mean that \( B = C \)
- \( AB = 0 \) does not mean that \( A \) or \( B \) is 0
Sums and Scalar Multiples

Same as for vectors: component-wise, so matrices must be same size to add.

\[ A + B = B + A \]

\[ (A + B) + C = A + (B + C) \]

\[ r(A + B) = rA + rB \]

\[ (r + s)A = rA + sA \]

\[ (rs)A = r(sA) \]

\[ A + 0 = A \]

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

(We can define linear transformations \( T + U \) and \( cT \), and so all of the above facts are also facts about linear transformations.)
Summary of Section 3.4

- Composition: \((T \circ U)(v) = T(U(v))\) (do \(U\) then \(T\))
- Matrix multiplication: \((AB)_{ij} = r_i \cdot b_j\)
- Matrix multiplication: the \(i\)th column of \(AB\) is \(A(b_i)\)
- Suppose that \(A\) and \(B\) are the standard matrices for the linear transformations \(T : \mathbb{R}^n \rightarrow \mathbb{R}^m\) and \(U : \mathbb{R}^p \rightarrow \mathbb{R}^n\). The standard matrix for \(T \circ U\) is \(AB\).
- **Warning!**
  - \(AB\) is not always equal to \(BA\)
  - \(AB = AC\) does not mean that \(B = C\)
  - \(AB = 0\) does not mean that \(A\) or \(B\) is 0
Section 3.5 Outline

- The definition of a matrix inverse
- How to find a matrix inverse
- Inverses for linear transformations

\[ 7x = 35 \]
\[ 7^{-1} \cdot 7x = 7^{-1} \cdot 35 \]
\[ 1 \cdot x = 5 \]
Inverses

To solve

\[ Ax = b \]

we might want to “divide both sides by \( A \)”.

We will make sense of this...
Inverses

$A = n \times n$ matrix.

$A$ is invertible if there is a matrix $B$ with

$$AB = BA = I_n$$

$B$ is called the inverse of $A$ and is written $A^{-1}$

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
The $2 \times 2$ Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then \( \text{det}(A) = ad - bc \) is the determinant of $A$.

**Fact.** If \( \text{det}(A) \neq 0 \) then $A$ is invertible and $A^{-1} = \frac{1}{\text{det}(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If \( \text{det}(A) = 0 \) then $A$ is not invertible.

\[
\begin{pmatrix} 5 & 10 \\ 7 & 14 \end{pmatrix} = 5 \cdot 14 - 7 \cdot 10 = 0
\]

not invertible.

**Example.** \[
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.
\]

\[
-\frac{1}{2} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
Solving Linear Systems via Inverses

**Fact.** If $A$ is invertible, then $Ax = b$ has exactly one solution:

$$ x = A^{-1} b. $$

Solve

$$
\begin{align*}
2x + 3y + 2z &= 1 \\
x + 3z &= 1 \\
2x + 2y + 3z &= 1
\end{align*}
$$

Using

$$
\begin{pmatrix}
2 & 3 & 2 \\
1 & 0 & 3 \\
2 & 2 & 3
\end{pmatrix}^{-1} =
\begin{pmatrix}
-6 & -5 & 9 \\
3 & 2 & -4 \\
2 & 2 & -3
\end{pmatrix}
$$
Solving Linear Systems via Inverses

What if we change \( b \)?

\[
\begin{align*}
2x + 3y + 2z &= 1 \\
x + 3z &= 0 \\
2x + 2y + 3z &= 1
\end{align*}
\]

Using

\[
\begin{pmatrix}
2 & 3 & 2 \\
1 & 0 & 3 \\
2 & 2 & 3
\end{pmatrix}^{-1} = \begin{pmatrix}
-6 & -5 & 9 \\
3 & 2 & -4 \\
2 & 2 & -3
\end{pmatrix}
\]

So finding the inverse is essentially the same as solving all \( Ax = b \) equations at once (fixed \( A \), varying \( b \)).