Announcements Jan 29

- Midterm 1 on Friday
- WeBWorK 2.3, 2.4 & 2.5 due Thursday
- My office hours Monday 3-4 and Wed 2-3
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems and practice exams on the master web site
  on calendar
Section 2.5
Linear Independence
Section 2.5 Outline

- Understand what it means for a set of vectors to be linearly independent
- Understand how to check if a set of vectors is linearly independent
Linear Independence

*Basic question:* What is the smallest number of vectors needed in the parametric solution to a linear system?

A set of vectors \( \{v_1, \ldots, v_k\} \) in \( \mathbb{R}^n \) is **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0
\]

has only the **trivial solution.** It is **linearly dependent** otherwise. \( \Rightarrow \) all \( x_i = 0 \)

So, linearly dependent means there are \( x_1, x_2, \ldots, x_k \) not all zero so that

\[
x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0
\]

This is a *linear dependence* relation.
Linear Independence

A set of vectors \( \{v_1, \ldots, v_k \} \) in \( \mathbb{R}^n \) is **linearly independent** if the vector equation

\[
 x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0
\]

has only the trivial solution.

**Fact.** The columns of \( A \) are linearly independent
\[\iff Ax = 0 \text{ has only the trivial solution.}\]
\[\iff A \text{ has a pivot in each column}\]

Why?
Linear Independence

Is \( \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ 4 \end{pmatrix} \right\} \) linearly independent?

\[ 2 \cdot \sqrt{1} + 1 \cdot \sqrt{2} = \sqrt{3} \]

\[ \text{lin. dependence} \]

\[ X_3 = -1 \]

\[ 3 \text{ pivots} \]

\[ \text{lin. ind.} \]
Linear Independence

When is \( \{v\} \) is linearly dependent?

When is \( \{v_1, v_2\} \) is linearly dependent? \[ V = 0 \]

When is the set \( \{v_1, v_2, \ldots, v_k\} \) linearly dependent?

**Fact.** The set \( \{v_1, v_2, \ldots, v_k\} \) is linearly dependent if and only if they span a \( k \)-dimensional plane.

**Fact.** The set \( \{v_1, v_2, \ldots, v_k\} \) is linearly dependent if and only if we can remove a vector from the set without changing the dimension of the span.

**Fact.** The set \( \{v_1, v_2, \ldots, v_k\} \) is linearly dependent if and only if some \( v_i \) lies in the span of \( v_1, \ldots, v_{i-1} \).
Span and Linear Independence

Is \{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \} \text{ linearly independent?}

\begin{align*}
V_1 \text{ row red } & \rightarrow 3 \text{ pivots } \rightarrow \text{ indep.} \\
V_2 \text{ is not } 0. \\
V_2 \text{ is not mult. of } V_1, \\
& \text{ (not a lin comb. of } V_1) \\
V_3 \text{ is not a lin comb of } V_1, V_2 \\
& \text{ because } V_1, V_2 \text{ lie in xy plane} \\
V_3 \text{ does not.}
\end{align*}
Linear independence and free variables

**Theorem.** Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and consider the vector equation

$$x_1v_1 + \cdots + x_kv_k = 0.$$ 

The set of vectors corresponding to non-free variables are linearly independent.

So, given a bunch of vectors $v_1, \ldots, v_k$, if you want to find a collection of $v_i$ that are linearly independent, you put them in the columns of a matrix, row reduce, find the pivots, and then take the original $v_i$ corresponding to those columns.

**Example.** Try this with $(1, 1, 1)$, $(2, 2, 2)$, and $(1, 2, 3)$.

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{pmatrix}
\xrightarrow{\text{pivot cols}}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$$
Linear independence and coordinates

**Fact.** If $v_1, \ldots, v_k$ are linearly independent vectors then we can write each element of

$$\text{Span}\{v_1, \ldots, v_k\}$$

in exactly one way as a linear combination of $v_1, \ldots, v_k$.

More on this later, when we get to bases.
Span and Linear Independence

Two More Facts

Fact 1. Say $v_1, \ldots, v_k$ are in $\mathbb{R}^n$. If $k > n$, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0
\end{pmatrix}
\]

Fact 2. If one of $v_1, \ldots, v_k$ is 0, then $\{v_1, \ldots, v_k\}$ is linearly dependent.
Parametric vector form and linear independence

Say you find the parametric vector form for a homogeneous system of linear equations, and you find that the set of solutions is the span of certain vectors. Then those vectors are...

1. always linearly independent
2. sometimes linearly independent
3. never linearly independent

Example

\[ x_1 + x_2 + x_3 = 0 \]

\[
\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\]

These two...
Parametric Vector Forms and Linear Independence

In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors that appear are linearly independent (why?). This means that we can't write the solution with fewer than two vectors (why?). This also means that this way of writing the solution set is efficient: for each solution, there is only one choice of $x_3$ and $x_4$ that gives that solution.
Summary of Section 2.5

- A set of vectors \( \{v_1, \ldots, v_k\} \) in \( \mathbb{R}^n \) is **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0
\]

has only the trivial solution. It is **linearly dependent** otherwise.

- The cols of \( A \) are linearly independent
  \[\iff Ax = 0 \text{ has only the trivial solution.}\]
  \[\iff A \text{ has a pivot in each column}\]

- The number of pivots of \( A \) equals the dimension of the span of the columns of \( A \)

- The set \( \{v_1, \ldots, v_k\} \) is linearly independent \(\iff\) they span a \( k \)-dimensional plane

- The set \( \{v_1, \ldots, v_k\} \) is linearly dependent \(\iff\) some \( v_i \) lies in the span of \( v_1, \ldots, v_{i-1} \).

- To find a collection of linearly independent vectors among the \( \{v_1, \ldots, v_k\} \)