Announcements Jan 29

- Midterm 1 on Friday
- WeBWorK 2.3, 2.4 & 2.5 due Thursday
- My office hours Monday 3-4 and Wed 2-3
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel see email
- Supplemental problems and practice exams on the master web site

on calendar

Section 2.5

Linear Independence

Section 2.5 Outline

- Understand what is means for a set of vectors to be linearly independent
- Understand how to check if a set of vectors is linearly independent

Linear Independence

Basic question: What is the smallest number of vectors needed in the parametric solution to a linear system?

A set of vectors $\{v_1, \dots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

has only the trivial solution. It is linearly dependent otherwise.

all $x_i = 0$

example 1

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$2 \cdot v_1 + (-1) \cdot v_2 = 0$

So, linearly dependent means there are $x_1, x_2, \dots, x_k$ not all zero so that

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

This is a *linear dependence* relation.

example 2

$v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$1 \cdot v_1 + 0 \cdot v_2 = 0$

Linear Independence

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

has only the trivial solution.

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 2 \\ 0 & 0 \end{pmatrix}$$

Fact. The columns of A are linearly independent
$\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
$\Leftrightarrow$ A has a pivot in each column

Why?

Linear Independence

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

$x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 0$

row reduce $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\rightsquigarrow$ 2 pivots

$2 \cdot v_1 + 1 \cdot v_2 = v_3$

lin. dependence

$x_1 = -2x_3$

$x_2 = -x_3$

$x_3 = x_3$

$x_3 = -1$

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

3 pivots

$\rightsquigarrow$ lin. ind.

Linear Independence

When is $\{v\}$ is linearly dependent? only when $v = 0$

$\begin{pmatrix}1\\2\\3\end{pmatrix} \begin{pmatrix}2\\4\\6\end{pmatrix} \begin{pmatrix}1\\2\\3\end{pmatrix} \begin{pmatrix}1\\1\\1\end{pmatrix}$

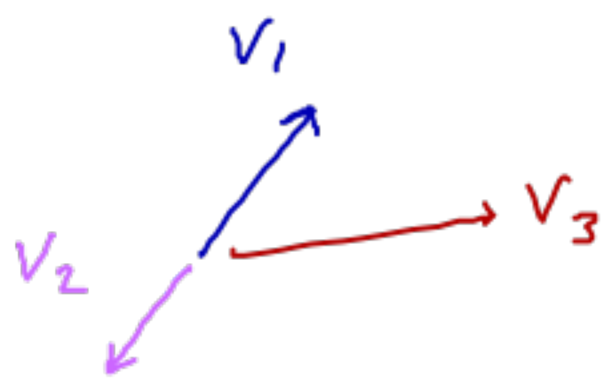
When is $\{v_1, v_2\}$ is linearly dependent? v_1 is scalar mult. of v_2 $\iff$ lie on same line.

When is the set $\{v_1, v_2, \ldots, v_k\}$ linearly dependent?

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly in dependent if and only if they span a k-dimensional plane.

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if we can remove a vector from the set without changing ~~the dimension of~~ the span.

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some v_i lies in the span of $v_1, \ldots, v_{i-1}$.



Demo

Span and Linear Independence

Is $\left\{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

$v_1 \quad v_2 \quad v_3$

row red $\rightsquigarrow$ 3 pivots $\rightsquigarrow$ indep.

v_1 is not 0.

v_2 is not mult. of v_1

(not a lin comb. of v_1)

v_3 is not a lin comb of v_1, v_2

because v_1, v_2 lie in xy plane

v_3 does not.

Linear independence and free variables

Theorem. Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and consider the vector equation

$$x_1 v_1 + \cdots + x_k v_k = 0.$$

The set of vectors corresponding to non-free variables are linearly independent.

So, given a bunch of vectors $v_1, \ldots, v_k$, if you want to find a collection of v_i that are linearly independent, you put them in the columns of a matrix, row reduce, find the pivots, and then take the original v_i corresponding to those columns.

Example. Try this with $(1, 1, 1)$, $(2, 2, 2)$, and $(1, 2, 3)$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 2 & \cancel{1}\,0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & \cancel{2}\,0 \end{pmatrix}$$

pivot cols

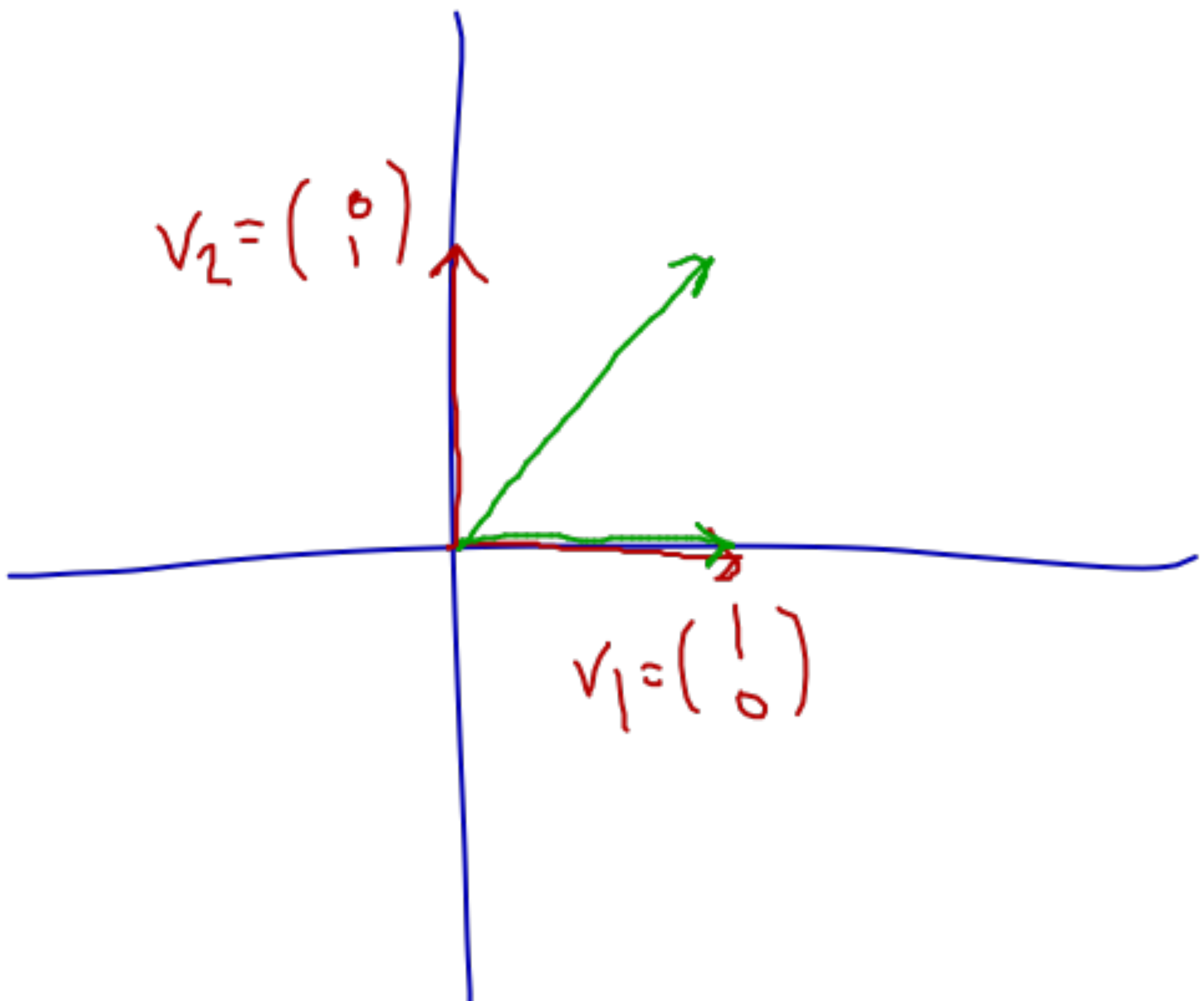
Linear independence and coordinates

Fact. If $v_1, \ldots, v_k$ are linearly independent vectors then we can write each element of

$$\mathrm{Span}\{v_1, \ldots, v_k\}$$

in exactly one way as a linear combination of $v_1, \ldots, v_k$.

More on this later, when we get to bases.



- $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3 \cdot v_1 + 1 \cdot v_2$

Span and Linear Independence

Two More Facts

Fact 1. Say $v_1, \ldots, v_k$ are in $\mathbb{R}^n$. If $k > n$, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Fact 2. If one of $v_1, \ldots, v_k$ is 0, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

Parametric vector form and linear independence

Poll

Say you find the parametric vector form for a homogeneous system of linear equations, and you find that the set of solutions is the span of certain vectors. Then those vectors are...

using our recipe.

1. always linearly independent
2. sometimes linearly independent
3. never linearly independent

these two

example

$$x_1 + x_2 + x_3 = 0$$

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{aligned} x_1 &= -x_2 - x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned} \longrightarrow x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Parametric Vector Forms and Linear Independence

In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors that appear are linearly independent (why?). This means that we can't write the solution with fewer than two vectors (why?). This also means that this way of writing the solution set is efficient: for each solution, there is only one choice of x_3 and x_4 that gives that solution.

Summary of Section 2.5

- A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

  has only the trivial solution. It is linearly dependent otherwise.
- The cols of A are linearly independent
  $\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
  $\Leftrightarrow$ A has a pivot in each column
- The number of pivots of A equals the dimension of the span of the columns of A
- The set $\{v_1, \ldots, v_k\}$ is linearly independent $\Leftrightarrow$ they span a k-dimensional plane
- The set $\{v_1, \ldots, v_k\}$ is linearly dependent $\Leftrightarrow$ some v_i lies in the span of $v_1, \ldots, v_{i-1}$.
- To find a collection of linearly independent vectors among the $\{v_1, \ldots, v_k\}$