Announcements Feb 4

- Midterm 1 on Friday in studio
- WeBWorK 2.3, 2.4 & 2.5 due Thursday
- My office hours Monday 3-4 and Wed 2-3
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- Review sessions
  - Kalen 7 pm Thu `https://bluejeans.com/5404873994`
  - Kyle Skiles 154 Wed 5-7
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel (different this week) 5-7 today Clough 152
- Supplemental problems and practice exams on the master web site

Section 2.6

Subspaces

Outline of Section 2.6

- Definition of subspace
- Examples and non-examples of subspaces
- Spoiler alert: Subspaces are the same as spans
- Spanning sets for subspaces
- Two important subspaces for a matrix: $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$

Subspaces

A subspace of $\mathbb{R}^n$ is a subset V of $\mathbb{R}^n$ with:

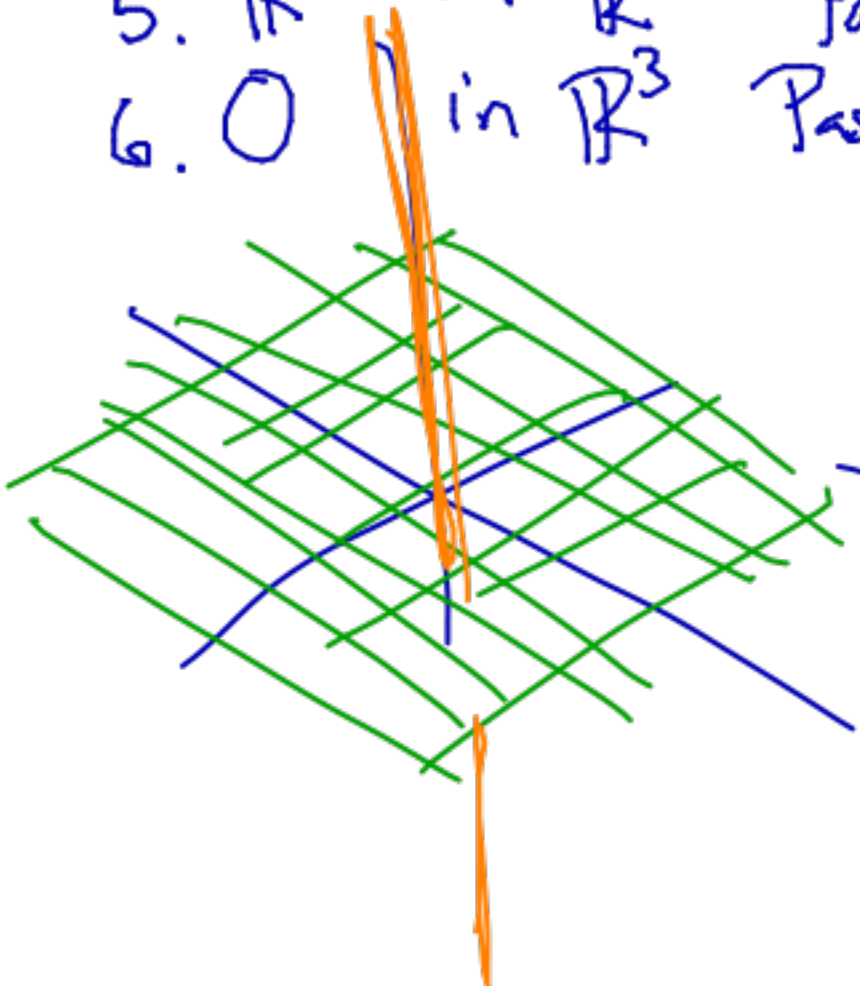
1. The zero vector is in V.
2. If u and v are in V, then $u + v$ is also in V.
3. If u is in V and c is a scalar, then cu is in V.

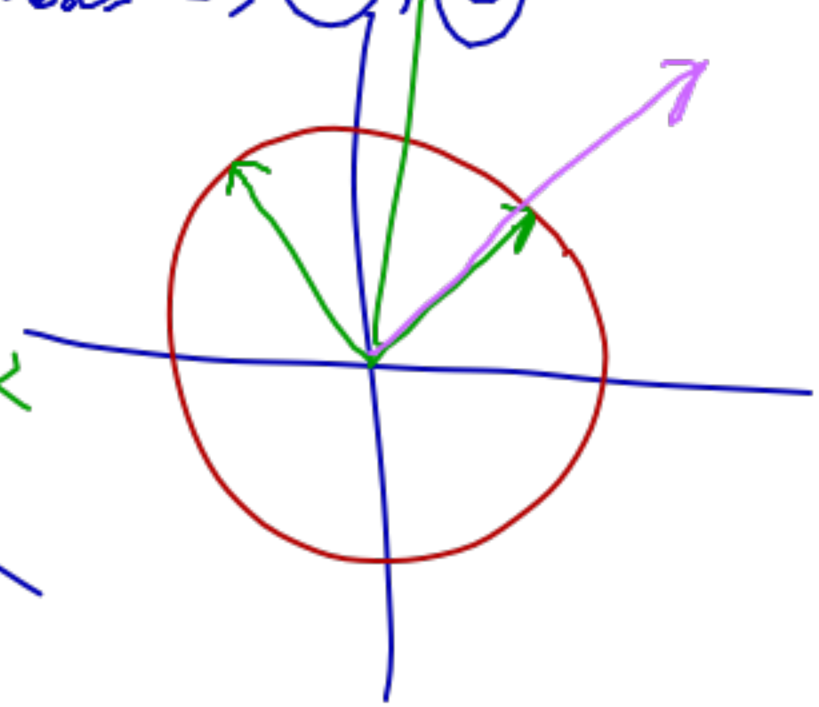
The second and third properties are called "closure under addition" and "closure under scalar multiplication."

Together, the second and third properties could together be rephrased as: closure under linear combinations.

Which are subspaces?

1. the unit circle in $\mathbb{R}^2$ No. Fails 1 2 3
2. the point $(1,2,3)$ in $\mathbb{R}^3$ No. Fails 1 2 3
3. the xy-plane in $\mathbb{R}^3$ Passes 1, 2, 3 yes!
4. the xy-plane together with the z-axis in $\mathbb{R}^3$ Passes 1, 3 Fails 2: $\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$
5. $\mathbb{R}^3$ in $\mathbb{R}^3$ Passes 1, 2, 3
6. 0 in $\mathbb{R}^3$ Passes 1, 2, 3



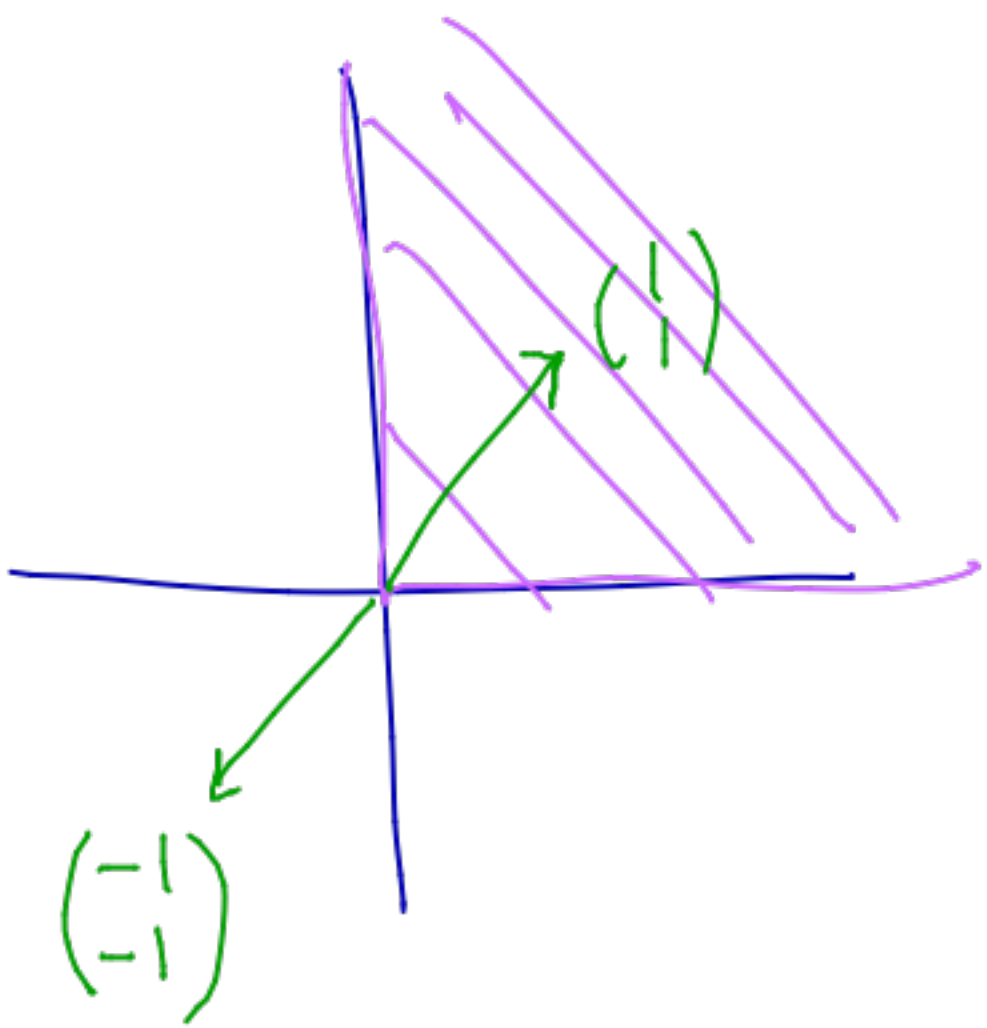


Which are subspaces?

Poll

Is the first quadrant of $\mathbb{R}^2$ a subspace?

1. yes
2. no



1 yes

2 yes $\begin{pmatrix}5\\0\end{pmatrix}+\begin{pmatrix}7\\9\end{pmatrix}=\cdots$

3 no, scalar mult. by -1.

Which are subspaces?

the set of ... so that

1. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \mid a+b=0 \right\}$

① ✓

② ✓ $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \end{pmatrix}$

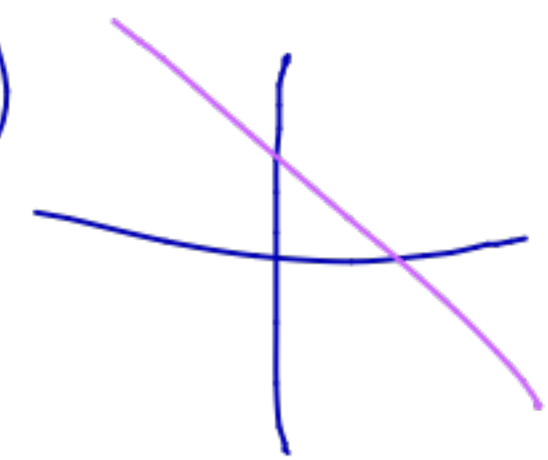
$\begin{pmatrix} 7 \\ -7 \end{pmatrix} + \begin{pmatrix} 6 \\ -6 \end{pmatrix}$

③ ✓

2. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \mid a+b=1 \right\}$

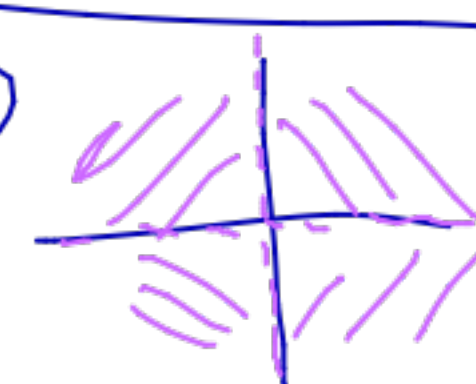
$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ldots$

Fails ①



3. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \mid ab \neq 0 \right\}$

Fails ①



4. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \mid a, b \text{ rational} \right\}$

① ✓

② ✓

③ × $\pi \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix}$

Spans and subspaces

① : $0 \cdot v_1 + 0 \cdot v_2$

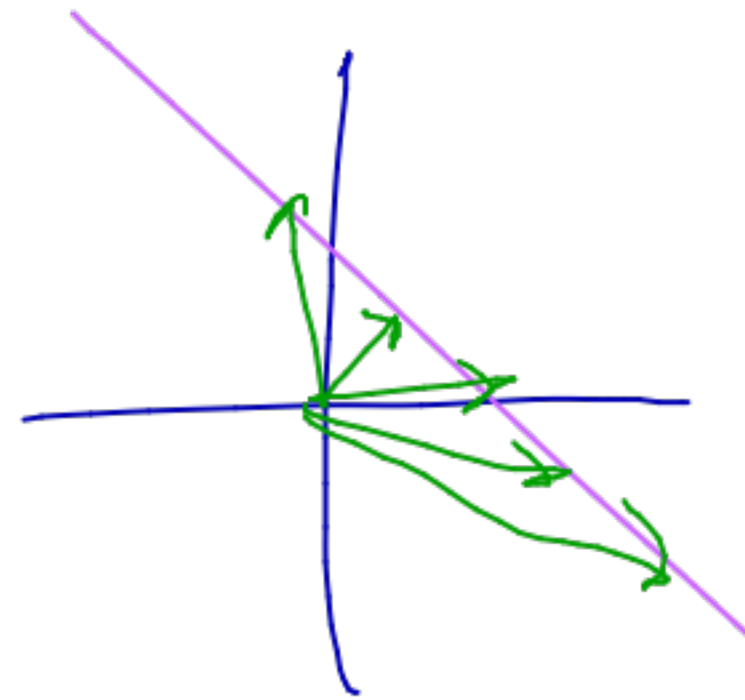
Fact. Any $\mathrm{Span}\{v_1, \ldots, v_k\}$ is a subspace.

Why? ② : $(5v_1 + 3v_2) + (v_1 - v_2) = 6v_1 + 2v_2$

Fact. Every subspace V of $\mathbb{R}^n$ is a span.

Why?

V = Span of all vectors in V.



So now we know that three things are the same:

- subspaces
- spans
- planes through 0

So why bother with the word "subspace"? Sometimes easier to check a subset is a subspace than to check it is a span (see null spaces, eigenspaces). Also, it makes sense (and is often useful) to think of a subspace *without a particular spanning set in mind*.

Column Space and Null Space

$A = m \times n$ matrix.

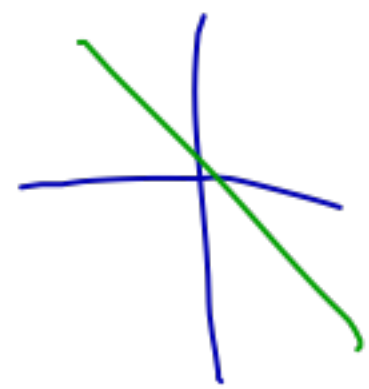
$\mathrm{Col}(A)$ = column space of A = span of the columns of A

$\mathrm{Nul}(A)$ = null space of A = (set of solutions to $Ax = 0$)

Example. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$ $x+y=0$

Col(A) = line in $\mathbb{R}^3$

Nul(A) line in $\mathbb{R}^2$



$\mathrm{Col}(A)$ = subspace of $\mathbb{R}^m$ new!

$\mathrm{Nul}(A)$ = subspace of $\mathbb{R}^n$

find by row red as usual

Why is Nul(A) a subspace?

(1) ✓

(2) Say v, w in Nul A:

So $Av = 0$ & $Aw = 0$

So $A(v+w) = Av + Aw = 0 + 0 = 0$.

(3) similar.

Already knew Nul(A) is a span (param. vect. form)

Spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\underline{\mathrm{Col}(A)}$: $\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ line in $\mathbb{R}^3$

$\underline{\mathrm{Nul}(A)}$: $x+y+z=0$ plane in $\mathbb{R}^3$

$x=-y-z$

$y=y$

$z=z$

$\rightsquigarrow \left\{ y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : y,z \text{ real nums} \right\}$

$$\mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a spanning set for $\mathrm{Nul}(A)$
- the pivot columns of A form a spanning set for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

$$A = \begin{pmatrix} \boxed{\begin{matrix}1\\2\\3\end{matrix}} & \begin{matrix}2\\4\\6\end{matrix} & \begin{matrix}3\\6\\9\end{matrix} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that the columns of A form a (possibly larger) spanning set. We'll see later that the above recipe is the smallest spanning set.

Spanning sets

Find a spanning set for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

Want Null space of

$$\begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3/2 & 1/2 \end{pmatrix}$$

$x = -3/2\, y - 1/2\, z$

$y = y$

$z = z$

$$\longrightarrow y\begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Span}\left\{\begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Subspaces and Null spaces

Fact. Every subspace is a null space.

Why? Given a spanning set, you can reverse engineer the A...

It's actually a little tricky to do this. Given the spanning set, you make those vectors the rows of a matrix, then row reduce, and then the rows of the reduced matrix are vectors that look like the kinds of vectors we get from vector parametric form. Why does this work? Try an example!

So now we know that four things are the same:

- subspaces
- spans
- planes through 0
- solutions to $Ax = 0$

So why learn about subspaces?

If subspaces are the same as spans, planes through the origin, and solutions to $Ax = 0$, why bother with this new vocabulary word?

The point is that we have been throwing around terms like "3-dimensional plane in $\mathbb{R}^4$" all semester, but we never said what "dimension" and "plane" are. Subspaces give the proper way to define a plane. Soon we will learn the meaning of a dimension of a subspace.

Section 2.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u + v$ is also in V.
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.
- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- Four things are the same: subspaces, spans, planes through 0, null spaces