Announcements Jan 15

- WeBWorK due Thursday
- Quiz in studio on Friday
- My office hours Monday 3-4 and Wed 2-3 in Skiles 234
- TA Office Hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel (starting Wed)
- If you have a general question, post on Piazza
- I prefer direct email to Canvas email
- Keep your laptops put away

Section 1.2

Row reduction

Outline of Section 1.2

- Solve systems of linear equations via elimination
- Solve systems of linear equations via matrices and row reduction
- Learn about row echelon form and reduced row echelon form of a matrix
- Learn the algorithm for finding the (reduced) row echelon form of a matrix
- Determine from the row echelon form of a matrix if the corresponding system of linear equations is consistent or not.

Row Reduction and Echelon Forms

Row Reduction and Echelon Forms

A matrix is in row echelon form if

1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is easy to solve using back substitution.

The pivot positions are the leading entries in each row.

Reduced Row Echelon Form

A system is in reduced row echelon form if also:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

For example:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is even easier to solve.

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right) \quad \left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right) \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Can every matrix be put in reduced row echelon form?

Row Reduction

Theorem. Each matrix is row equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm. That shows a matrix is equivalent to at least one matrix in reduced row echelon form.

Row Reduction Algorithm

To find row echelon form:

Step 1 Swap rows so a leftmost nonzero entry is in 1st row (if needed)

Step 2 Scale 1st row so that its leading entry is equal to 1

Step 3 Use row replacement so all entries below this 1 (or, pivot) are 0

Then cover the first row and repeat the three steps.

To then find reduced row echelon form:

- Use row replacement so that all entries above the pivots are 0.

Examples.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \left(\begin{array}{ccc|c} 0 & 7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right) \left(\begin{array}{ccc|c} 4 & -5 & 3 & 2 \\ 1 & -1 & -2 & -6 \\ 4 & -4 & -14 & 18 \end{array}\right)$$

▸ Interactive Row Reducer

Solutions of Linear Systems

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

What are the solutions? Say the variables are x and y.

$x = 5$

$y = 2$

Solutions of Linear Systems: Consistency

Solve the linear system associated to:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Say the variables are x, y, and z.

$0 = 1$

$\Longrightarrow$ no solution

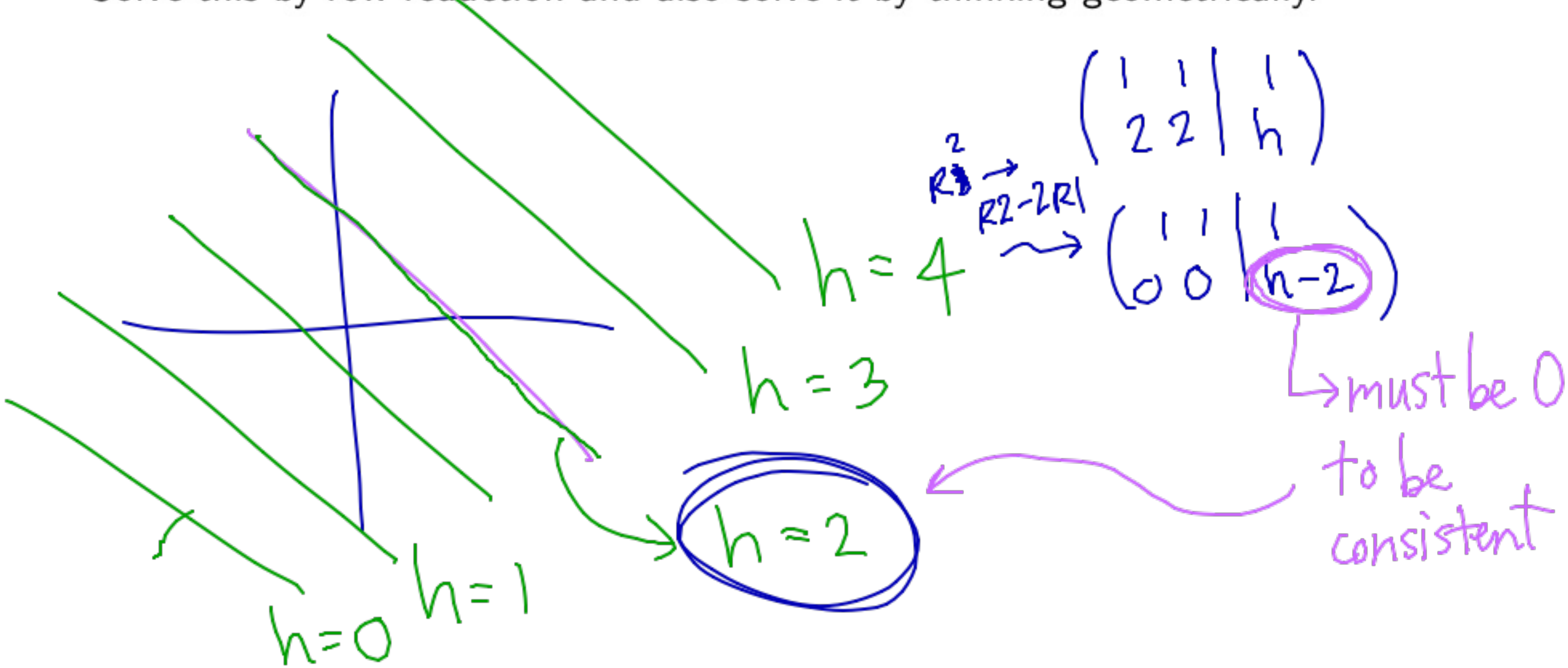
A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

Example with a parameter

For which values of h does the following system have a solution?

$$x + y = 1$$
$$2x + 2y = h$$

Solve this by row reduction and also solve it by thinking geometrically.



Summary of Section 1.2

- To solve a system of linear equations we can use the method of elimination.
- We can more easily do elimination with matrices. The allowable moves are row swaps, row scales, and row replacements. This is called row reduction.
- A matrix in row echelon form corresponds to a system of linear equations that we can easily solve by back substitution.
- A matrix in reduced row echelon form corresponds to a system of linear equations that we can easily solve just by looking.
- We have an algorithm for row reducing a matrix to row echelon form.
- The reduced row echelon form of a matrix is unique.
- Two matrices that differ by row operations are called row equivalent.
- A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

1.3 Parametric Form

Outline of Section 1.3

- Find the parametric form for the solutions to a system of linear equations.
- Describe the geometric picture of the set of solutions.

Free Variables

We know how to understand the solution to a system of linear equations when every column to the left of the vertical line has a pivot. For instance:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

If the variables are x and y what are the solutions?

Free Variables

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? For instance:

no pivot

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 5 & 0 \\ 0 & \boxed{1} & 2 & 1 \end{array}\right)$$

represents two equations:

$$\begin{aligned} x_1 \qquad + 5x_3 &= 0 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

There is one free variable x_3, corresponding to the non-pivot column. To solve, we move the free variable to the right:

$$\begin{aligned} x_1 \qquad &= -5x_3 \\ x_2 &= 1 - 2x_3 \\ x_3 &= x_3 \text{ (free; any real number)} \end{aligned}$$

$(0,1,0)$ $X_3=0$

$(-250,-99,50)$ $X_3=50$

This is the parametric solution. We can also write the solution as:

$$(-5x_3, 1 - 2x_3, x_3)$$

What is one particular solution? What does the set of solutions look like?

Free Variables

Solve the system of linear equations in x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 \quad + 5x_3 \quad &= 0 \\ x_4 &= 0 \end{aligned}$$

So the associated matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

To solve, we move the free variable to the right:

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= \quad x_2 \quad \text{(free)} \\ x_3 &= \quad x_3 \quad \text{(free)} \\ x_4 &= 0 \end{aligned}$$

$(-15, 5, 3, 0)$

Or: $(-5x_3, x_2, x_3, 0)$. This is a plane in $\mathbb{R}^4$.

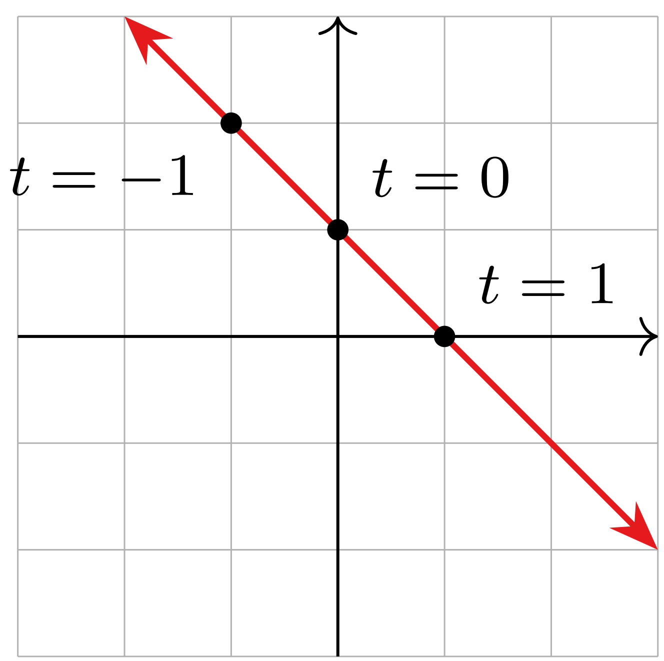
The original equations are the implicit equations for the solution. The answer to this question is the parametric solution.

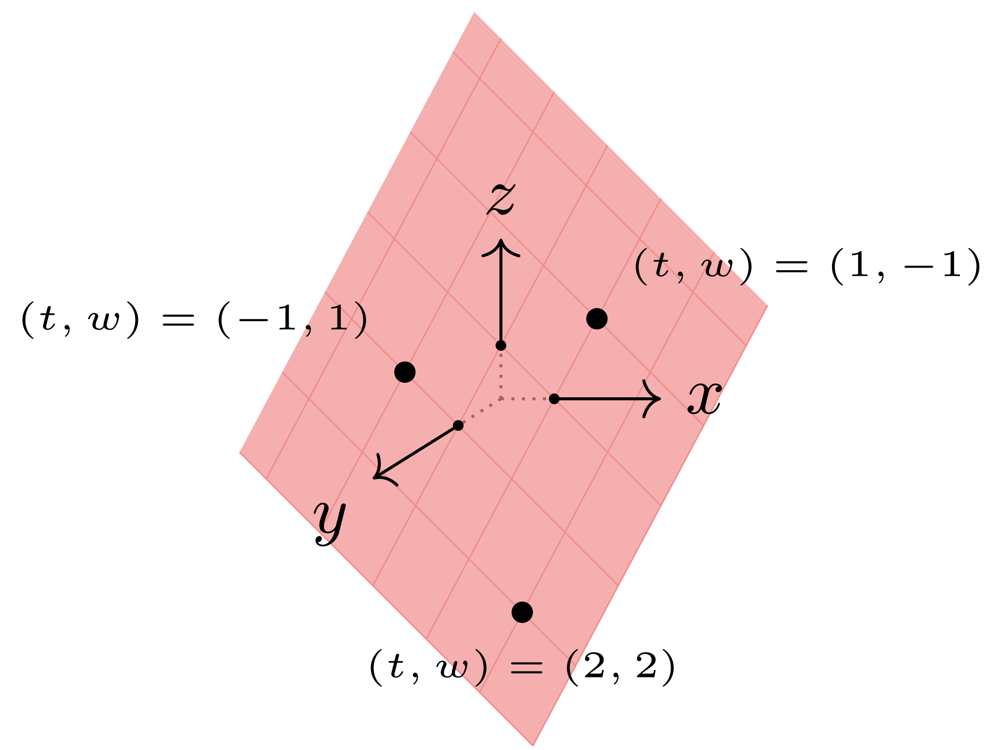
Free variables

Geometry

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional plane in $\mathbb{R}^n$.

Why does this make sense?





Poll

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. two points
4. line
5. plane
6. 3-dimensional plane
7. 4-dimensional plane

Implicit versus parametric equations of planes

Find a parametric description of the plane

$$x + y + z = 1$$

The original version is the implicit equation for the plane. The answer to this problem is the parametric description.

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of system of linear equations.

1. The last column is a pivot column.
   $\rightsquigarrow$ the system is *inconsistent*.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   $\rightsquigarrow$ the system has a *unique solution*.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   $\rightsquigarrow$ the system has *infinitely many* solutions; free variables correspond to columns without pivots.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$