Announcements Jan 22

- Midterm 1 on Feb 7
- WeBWorK due Thursday
- Quiz in studio on Friday
- My office hours Monday 3-4 and Wed 2-3 in Skiles 234
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel

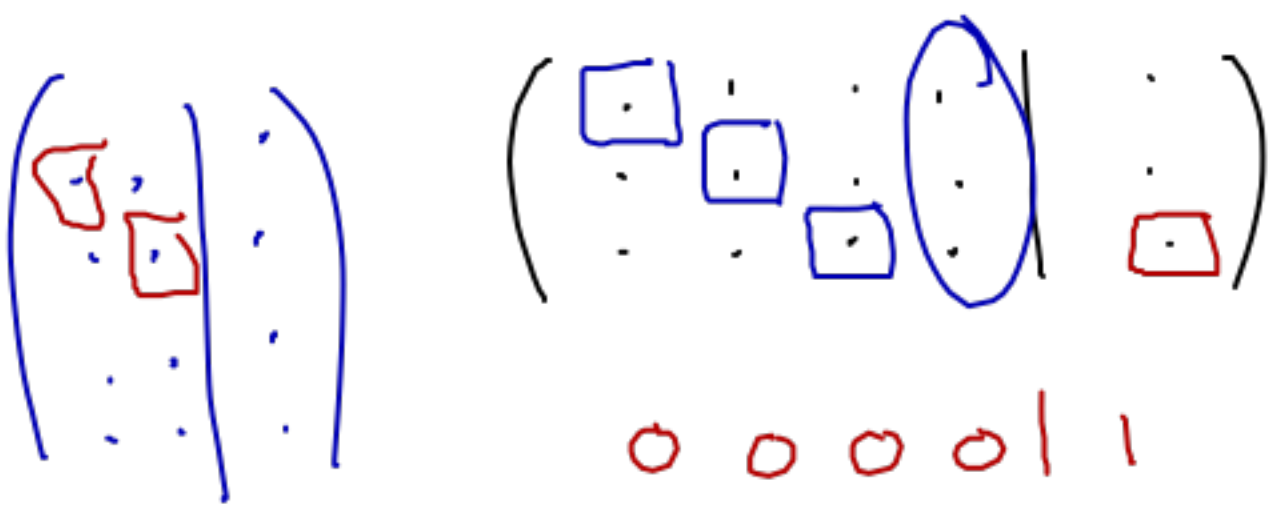
1.3 Parametric Form

Poll

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point — 0-dim plane
3. two points
4. line
5. plane
6. 3-dimensional plane
7. 4-dimensional plane — 4 free vars ⟷ 0 pivots ⟷ zero matrix

3 3-dim planes in $\mathbb{R}^4$
intersect them



pivots = 0, 1, 2, or 3

free vars = 4, 3, 2, 1

Implicit versus parametric equations of planes

Find a parametric description of the plane

$$x + y + z = 1$$

RREF

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right)$$

$$x = 1 - y - z$$
$$y = y$$
$$z = z$$

$$(1-y-z, y, z)$$

example

$y=5 \quad z=7$

$(-11, 5, 7)$ etc.

The original version is the implicit equation for the plane. The answer to this problem is the parametric description.

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of system of linear equations.

1. The last column is a pivot column.
   $\rightsquigarrow$ the system is *inconsistent*.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   $\rightsquigarrow$ the system has a *unique solution*.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   $\rightsquigarrow$ the system has *infinitely many* solutions; free variables correspond to columns without pivots.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

plane in $\mathbb{R}^4$

Chapter 2

System of Linear Equations: Geometry

Section 2.1

Vectors

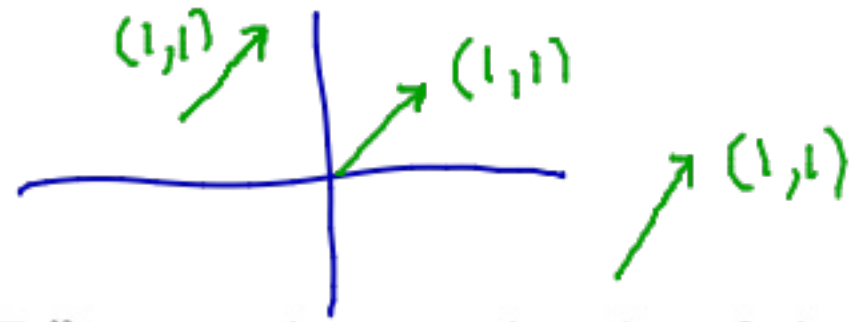
Outline

- Think of points in $\mathbb{R}^n$ as vectors.
- Learn how to add vectors and multiply them by a scalar
- Understand the geometry of adding vectors and multiplying them by a scalar
- Understand linear combinations algebraically and geometrically

Vectors

A vector is a matrix with one row or one column. We can think of a vector with n rows as:

- a point in $\mathbb{R}^n$
- an arrow in $\mathbb{R}^n$



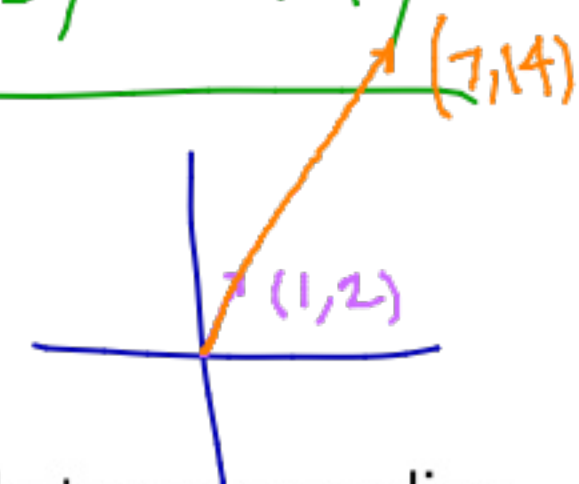
To go from an arrow to a point in $\mathbb{R}^n$, we subtract the tip of the arrow from the starting point. Note that there are many arrows representing the same vector.

Adding vectors / parallelogram rule ‣ Demo

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

Scaling vectors ‣ Demo

$$7 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$



A scalar is just a real number. We use this term to indicate that we are scaling a vector by this number.

Linear Combinations

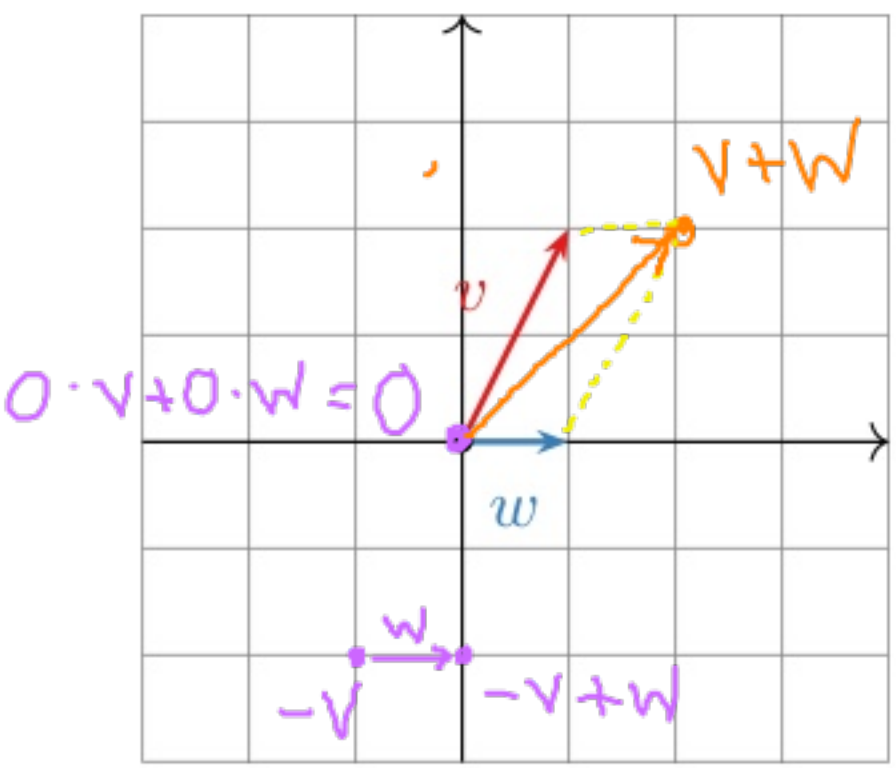
A linear combination of the vectors $v_1, \dots, v_k$ is any vector

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

where $c_1, \dots, c_k$ are real numbers.

$\cdot\ 2v + 3w$

identify $\begin{pmatrix} a \\ b \end{pmatrix}$ with (a,b) in $\mathbb{R}^2$



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

$$v + w = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$c_1 = 1 \quad c_2 = 1$

$1 \cdot v + 1 \cdot w$

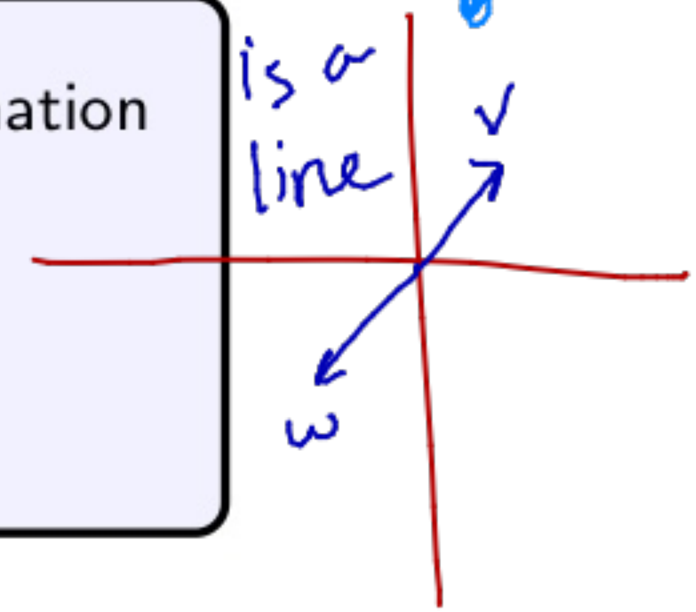
Span

Set of lin. combos

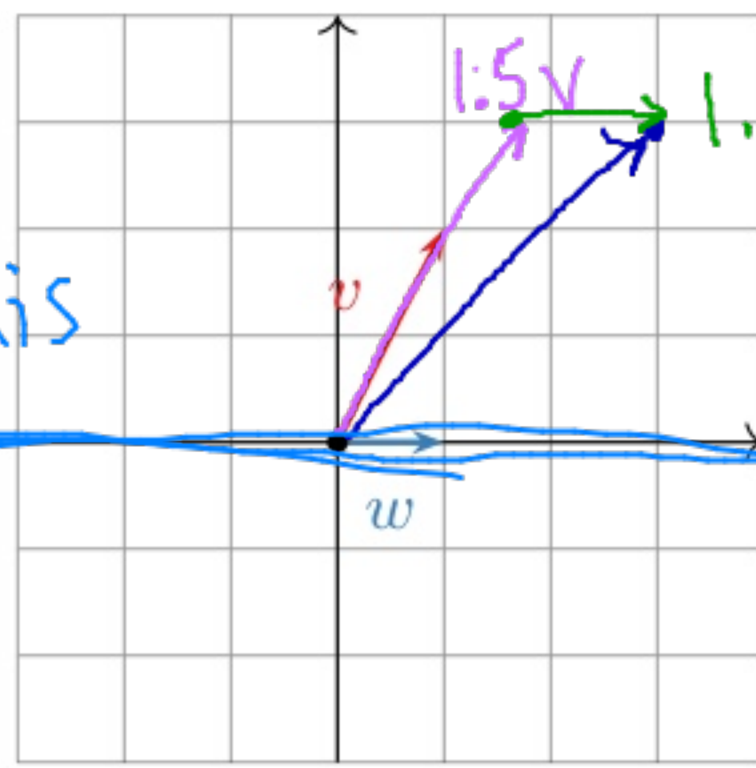
Poll

Is there a vector in $\mathbb{R}^2$ that is not a linear combination of v and w?

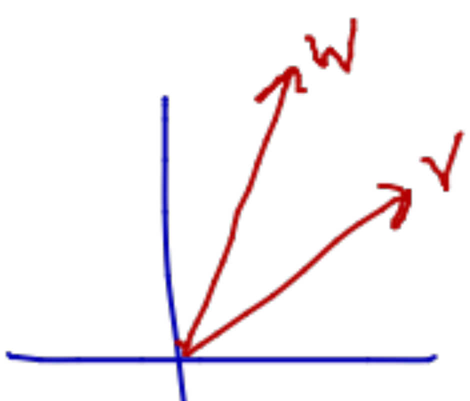
- yes
- no



Set of lin. combos of w = x-axis



$$1.5\begin{pmatrix}1\\2\end{pmatrix} + 1.5\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$$



Set of lin combos is $\mathbb{R}^2$

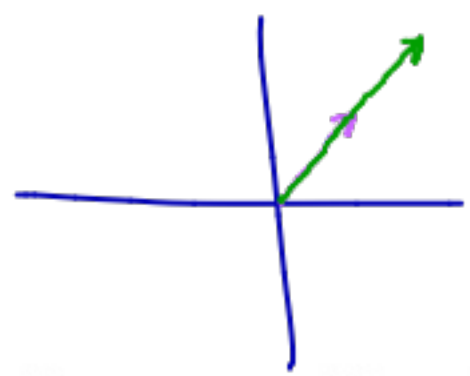
Set of linear combos of v & w is $\mathbb{R}^2$

Linear Combinations

What are some linear combinations of $(1,1)$?

(a,a) line y=x

What are some linear combinations of $(1,1)$ and $(2,2)$?



Same answer

What are some linear combinations of $(0,0)$?

origin

(0,0)

Summary of Section 2.1

- A vector is a point/arrow in $\mathbb{R}^n$
- We can add/scale vectors algebraically & geometrically (parallelogram rule)
- A linear combination of vectors $v_1, \ldots, v_k$ is a vector

  $$c_1 v_1 + \cdots + c_k v_k$$

  where $c_1, \ldots, c_k$ are real numbers.
- Asking the question of whether a certain vector is a linear combination of certain other vectors gives us a vector equation.
- Vector equations are the same as linear systems.

Section 2.2

Vector Equations and Spans

Outline of Section 3.2

- Learn the equivalences:

$$\text{vector equations} \leftrightarrow \text{augmented matrices} \leftrightarrow \text{linear systems}$$

- Learn the definition of span
- Learn the relationship between spans and consistency

Linear Combinations

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

vector eqn.

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

sys of lin eqns

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned}$$

row reduce!

Write down an equation in order to solve this problem. This is called a vector equation.

Notice that the vector equation can be rewritten as a system of linear equations. Solve it!

matrix eqn

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} \boxed{1} & -1 & 8 \\ 0 & \boxed{5} & -45 \\ 0 & 0 & 0 \end{array}\right)$$

Linear combinations, vector equations, and linear systems

In general, asking:

Is b a linear combination of $v_1, \ldots, v_k$?

is the same as asking if the vector equation

$$c_1 v_1 + \cdots c_k v_k = b$$

is consistent, which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).

Span

Essential vocabulary word!

$$\begin{aligned}\text{Span}\{v_1, v_2, \ldots, v_k\} &= \{c_1v_1 + c_2v_2 + \cdots c_kv_k \mid c_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

What are the possibilities for the span of two vectors in $\mathbb{R}^2$?

▸ Demo

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

▸ Demo

Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

Span

Essential vocabulary word!

$$\begin{aligned}\text{Span}\{v_1, v_2, \ldots, v_k\} &= \{c_1v_1 + c_2v_2 + \cdots c_kv_k \mid c_i \text{ in } \mathbb{R}\} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

Four ways of saying the same thing:

- b is in Span$\{v_1, v_2, \ldots, v_k\}$
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $c_1v_1 + \cdots + c_kv_k = b$ has a solution
- the system of linear equations corresponding to

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Demo Demo

Summary of Section 3.2

- vector equations $\leftrightarrow$ augmented matrices $\leftrightarrow$ linear systems
- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.
- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.