Announcements Jan 27

- Midterm 1 on Feb 7
- WeBWorK 2.1 & 2.2 due Thursday
- Quiz on 2.1 & 2.2 in studio on Friday
- My office hours Monday ~~2-3~~ 10-12 today (change only for today!) and Wed 2-3
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems on the master web site

Section 2.2

Vector Equations and Spans

Linear Combinations

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Write down an equation in order to solve this problem. This is called a vector equation.

$$\rightsquigarrow \begin{aligned} x_1 - x_2 &= 8 \\ 2x_1 - 2x_2 &= 16 \\ 6x_1 - x_2 &= 3 \end{aligned} \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right) \xrightarrow{\text{row reduce}}$$

$$x_1 = -1$$
$$x_2 = -9$$

Notice that the vector equation can be rewritten as a system of linear equations. Solve it!

Linear combinations, vector equations, and linear systems

In general, asking:

$$\text{Is } b \text{ a linear combination of } v_1, \ldots, v_k?$$

is the same as asking if the vector equation

$$c_1 v_1 + \cdots c_k v_k = b$$

is consistent, which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).

Span

Essential vocabulary word!

$$\begin{aligned}\text{Span}\{v_1, v_2, \ldots, v_k\} &= \{c_1v_1 + c_2v_2 + \cdots c_kv_k \mid c_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

What are the possibilities for the span of two vectors in $\mathbb{R}^2$?

Demo

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo

Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

Span

Essential vocabulary word!

$$\begin{aligned}\mathrm{Span}\{v_1, v_2, \ldots, v_k\} &= \{c_1 v_1 + c_2 v_2 + \cdots c_k v_k \mid c_i \text{ in } \mathbb{R}\} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

Four ways of saying the same thing:

- b is in Span$\{v_1, v_2, \ldots, v_k\}$
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $c_1 v_1 + \cdots + c_k v_k = b$ has a solution
- the system of linear equations corresponding to

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

  is consistent.

Demo Demo

Summary of Section 3.2

- vector equations $\leftrightarrow$ augmented matrices $\leftrightarrow$ linear systems
- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.
- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.

Section 2.3

Matrix equations

Outline Section 2.3

- Understand the equivalences:

  linear system $\leftrightarrow$ augmented matrix $\leftrightarrow$ vector equation $\leftrightarrow$ matrix equation

- Understand the equivalence:

  $$Ax = b \text{ is consistent} \longleftrightarrow b \text{ is in the span of the columns of } A$$

  (also: what does this mean geometrically)
- Learn for which A the equation $Ax = b$ is always consistent
- Learn to multiply a vector by a matrix

Multiplying Matrices

$$\text{matrix} \times \text{col vector} : \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 c_1 & + \cdots + & b_n c_n \\ | & & | \end{pmatrix}$$

Read this as: b_1 times the first column c_1 is the first column of the answer, b_2 times c_2 is the second column of the answer...

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 7\cdot 1 + 8\cdot 2 \\ 7\cdot 3 + 8\cdot 4 \\ 7\cdot 5 + 8\cdot 6 \end{pmatrix} = \begin{pmatrix} 7+16 \\ 21+32 \\ 35+48 \end{pmatrix}$$

$$= \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}$$

Multiplying Matrices

Another way to multiply

$$\text{row vector} \times \text{column vector} : \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$$

$$\text{matrix} \times \text{column vector} : \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 1\cdot 7 + 2\cdot 8 \\ 3\cdot 7 + 4\cdot 8 \\ 5\cdot 7 + 6\cdot 8 \end{pmatrix}$$

Same!

Linear Systems vs Augmented Matrices vs Matrix Equations vs Vector Equations

A matrix equation is an equation $Ax = b$ where A is a matrix and b is a vector. So x is a vector of variables.

A is an $m \times n$ matrix if it has m rows and n columns. What sizes must x and b be?

Example:

$$\boxed{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$$

$$\begin{aligned} x + 2y &= 9 \\ 3x + 4y &= 10 \\ 5x + 6y &= 11 \end{aligned}$$

Rewrite this equation as a vector equation, a system of linear equations, and an augmented matrix.

$$\left(\begin{array}{cc|c} 1 & 2 & 9 \\ 3 & 4 & 10 \\ 5 & 6 & 11 \end{array}\right) \qquad x\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$$

We will go back and forth between these four points of view over and over again. You need to get comfortable with this.

Solutions to Linear Systems vs Spans

Say that

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

Why?

Algebra

Geometry

Again this is a basic fact we will use over and over and over.

Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

incons.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

Soln: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Is a given vector in the span?

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

Is $(9, 10, 11)$ in the span of $(1, 3, 5)$ and $(2, 4, 6)$?

$$\begin{pmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}9\\ 10\\ 11\end{pmatrix}$$

$$\rightsquigarrow \left(\begin{array}{cc|c}1 & 2 & 9\\ 3 & 4 & 10\\ 5 & 6 & 11\end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c}1 & 2 & 9\\ 0 & -2 & -17\\ 0 & -4 & -34\end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c}\boxed{1} & 2 & 9\\ 0 & \boxed{-2} & -17\\ 0 & 0 & 0\end{array}\right)$$

consistent

$\rightsquigarrow$ yes!

Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,10,20)$, $(0,-1,-2)$
2. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,5,7)$, $(0,6,8)$
3. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,1,0)$, $(0,0,\sqrt{2})$
4. $(0,1,2)$ is in the span of $(5,7,0)$, $(6,8,0)$, $(3,3,4)$

$(1,2,3)$ $(2,4,6)$

$(1,2,4)$
not in span of $(3,6,9)$

Pivots vs Solutions

Theorem. Let A be an $\overset{\text{rows}}{m} \times \overset{\text{cols}}{n}$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b
2. The span of the columns of A is $\mathbb{R}^m$
3. A has a pivot in each row

Why?

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & ? \\ 1 & 2 & 3 & ? \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & ? \\ 0 & 0 & 0 & \boxed{} \end{array}\right)$$

if not 0,
incons.

More generally, if you have k vectors in $\mathbb{R}^n$ and you want to know the dimension of the span, you should row reduce and count the number of pivots.

Properties of the Matrix Product Ax

$7x = 35$

$c =$ real number, $u, v =$ vectors,

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

Application. If u and v are solutions to $Ax = 0$ then so is every element of $\mathrm{Span}\{u \cancel{+} , v\}$.

$$Au = 0$$
$$Av = 0$$
$$\Longrightarrow A(5u + 3v) = 5Au + 3Av = 0 + 0 = 0$$

Guiding questions

Here are the guiding questions for the rest of the chapter:

1. What are the solutions to $Ax = 0$?

2. For which b is $Ax = b$ consistent?

These are two separate questions!

Summary of Section 2.3

- Two ways to multiply a matrix times a column vector:

$$\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

OR

$$\begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 c_1 & \cdots & b_n c_n \\ | & & | \end{pmatrix}$$

- Linear systems, augmented matrices, vector equations, and matrix equations are all equivalent.
- Fact. $Ax = b$ has a solution $\Leftrightarrow$ b is in the span of columns of A
- Theorem. Let A be an $m \times n$ matrix. The following are equivalent.
  1. $Ax = b$ has a solution for all b
  2. The span of the columns of A is $\mathbb{R}^m$
  3. A has a pivot in each row