Announcements Mar 9

- Midterm 3 on April 10
- WeBWorK on Chapter 4 due Thursday
- No quiz on Friday (next quiz Mar 27)
- My office hours Monday 3-4 and Wed 2-3 in Skiles 234
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems and practice exams on the master web site

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \qquad \text{or}$$
> $$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), principal component analysis, Google, Netflix, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

A Question from Biology

In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year - think of it as a vector (f, s, t) - what is the population the next year?

$$\boxed{\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}}$$

$$\begin{cases} F_{2021} = 6 \cdot S_{2020} + 8 \cdot T_{2020} \\ S_{2021} = \frac{1}{2} F_{2020} \\ T_{2021} = \frac{1}{2} S_{2020} \end{cases}$$

$$10^{16} \times \begin{pmatrix} 2.25 \\ .56 \\ .14 \end{pmatrix}$$

$$A \to \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} F_N \\ S_N \\ T_N \end{pmatrix} = \begin{pmatrix} F_{N+1} \\ S_{N+1} \\ T_{N+1} \end{pmatrix}$$

$$10^{16} \begin{pmatrix} 2.45 \\ .61 \\ .15 \end{pmatrix}$$

50 yrs

$$\begin{pmatrix} 35 \\ 35 \\ 1.25 \end{pmatrix}$$

Now choose some starting population vector $u = (f, s, t)$ and choose some number of years ~~N~~ M. What is the new population after ~~N~~ M years?

$$10^{7} \times \begin{pmatrix} 2.88 \\ .72 \\ .18 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix}$$

Demo

$$A^{M} \begin{pmatrix} \text{orig.} \\ \text{pop.} \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$$

$$A \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 70 \\ 2.5 \\ 2.5 \end{pmatrix}$$

Chapter 5

Eigenvectors and eigenvalues

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

eigen $=$ *characteristic*

So Av points in the same direction as v.

This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

Examples

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix}, \quad \lambda = 2$$

rabbit pop. doubles each year

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 64 \\ 16 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix}$$

ratio of f, s, t rabbits

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4$$

$$\begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

How do you check?

$$Av = \lambda v$$
$$A(15v) = \lambda(15v)$$

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix}$$

Eigenvectors and Eigenvalues

Confirming eigenvectors

Poll

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \cancel{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$ are eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{yes} \quad \lambda = 2$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{yes} \quad \lambda = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{yes} \quad \lambda = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{no: } \begin{pmatrix} 3 \\ 3 \end{pmatrix} \text{ not a mult of } \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Check

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$. $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix}$

Want

$Av = 3v \quad v \neq 0$

$Av - 3v = 0$

$Av - 3Iv = 0$

$\underbrace{(A-3I)v = 0}$

Does null space of $A-3I$ have a nonzero vector?

Is $\mathrm{Nul}(A-3I) = 0$ or not?

$A - 3I =$

$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$

A nonzero vector in null space: $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

eigenvector

What is a general procedure for finding eigenvalues?

Eigenspaces

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

Why is this a subspace?

Fact. λ-eigenspace for $A = \mathrm{Nul}(A - \lambda I)$

Example. Find the eigenspaces for $\lambda = 2$ and $\lambda = -1$ and sketch.

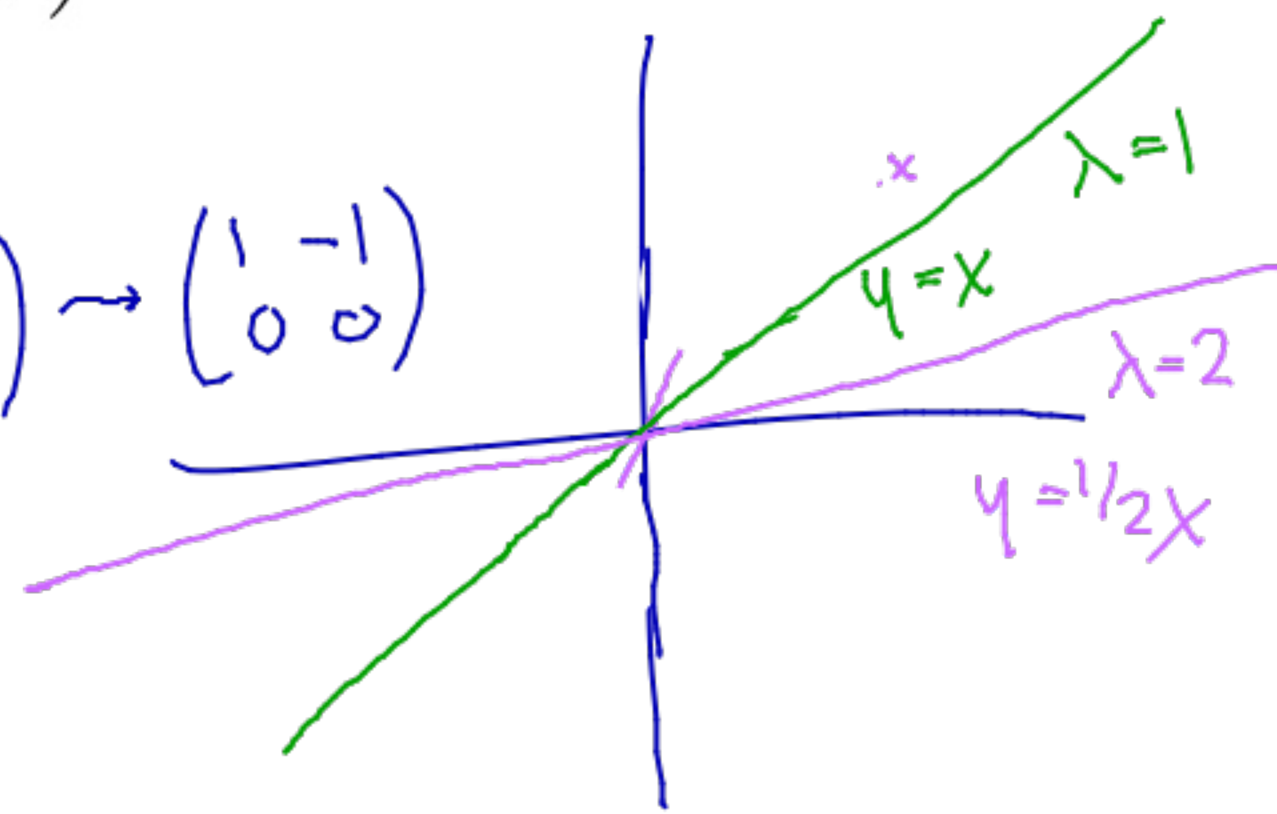
$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

$\underline{\lambda = 2}$

$$\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\underline{\lambda = -1}$

$$\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$



Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

Subtract 2 off the diag

$$\begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Find a basis for Null space

will be two of them.

$\rightsquigarrow$ 2 - Eigenspace is a plane.

Eigenvalues

And invertibility

Fact. A invertible $\Leftrightarrow 0$ is not an eigenvalue of A

Why?

λ eigenval

$\Longleftrightarrow \boxed{\mathrm{Nul}(A-\lambda I) \neq 0}$

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Fact. If $v_1 \ldots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots \lambda_k$, then $\{v_1, \ldots, v_k\}$ are linearly independent.

Why?

Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

- Reflection about the line $y = -x$ in $\mathbb{R}^2$
- Orthogonal projection onto the x-axis in $\mathbb{R}^2$
- Rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)
- Scaling of $\mathbb{R}^2$ by 3
- (Standard) shear of $\mathbb{R}^2$
- Orthogonal projection to the xy-plane in $\mathbb{R}^3$

Demo

Summary of Section 5.1

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- Recipe: The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A
- Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations

Review for Section 5.1

True or false: The zero vector is an eigenvector for every matrix.

What are the eigenvalues for a reflection about a line in $\mathbb{R}^2$?

How many different eigenvalues can there be for an $n \times n$ matrix?