Announcements Mar 2

- Midterm 2 on Friday
- I will be away March 3–6 (Jankowski will sub on Mar 4)
- WeBWorK 3.4, 3.5, and 3.6 due Thursday
- My office hours Monday 10:30-11:30 and 2-2:30 (this week only!) in Skiles 234
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- Review sessions TBA! Wed 5-7 CULC 152
- PLUS sessions Mon~~/Wed~~ 6-7 LLC West with Miguel
- Supplemental problems and practice exams on the master web site

Section 3.6

The invertible matrix theorem

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

(1) A is invertible

(2) T is invertible

(3) The reduced row echelon form of A is I_n

(4) A has n pivots

(5) $Ax = 0$ has only 0 solution

(6) $\mathrm{Nul}(A) = \{0\}$

(7) $\mathrm{nullity}(A) = 0$

(8) columns of A are linearly independent

(9) columns of A form a basis for $\mathbb{R}^n$

(10) T is one-to-one

(11) $Ax = b$ is consistent for all b in $\mathbb{R}^n$

(12) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$

(13) columns of A span $\mathbb{R}^n$

(14) $\mathrm{Col}(A) = \mathbb{R}^n$

(15) $\mathrm{rank}(A) = n$

(16) T is onto

(17) A has a left inverse

(18) A has a right inverse

$(A \mid I) \to (I \mid A^{-1})$

$T : \mathbb{R}^2 \to \mathbb{R}^2$

The Invertible Matrix Theorem

There are two kinds of square matrices, invertible and non-invertible matrices.

For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.

One way to think about the theorem is: there are lots of conditions equivalent to a matrix having a pivot in every row, and lots of conditions equivalent to a matrix having a pivot in every column, and when the matrix is a square, all of these many conditions become equivalent.

Example

Determine whether A is invertible. $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$

It isn't necessary to find the inverse. Instead, we may use the Invertible Matrix Theorem by checking whether we can row reduce to obtain three pivot columns, or three pivot positions.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

There are three pivot positions, so A is invertible by the IMT (statement c).

Where are we?

- We have studied the problem $Ax = b$
- We next want to study $Ax = \lambda x$
- At the end of the course we want to almost solve $Ax = b$

We need determinants for the second item.

Sections 4.1 and 4.3

The definition of the determinant and volumes

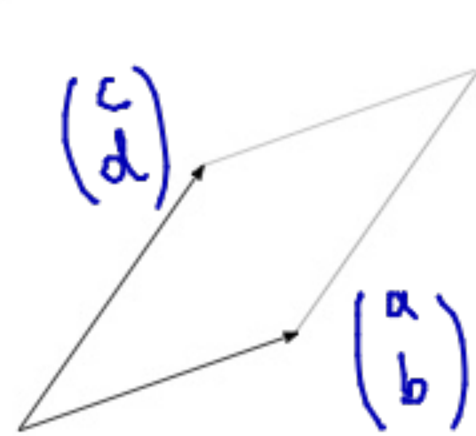
Outline of Sections 4.1 and 4.3

- Volume and invertibility
- A definition of determinant in terms of row operations
- Using the definition of determinant to compute the determinant
- Determinants of products: $\det(AB)$
- Determinants and linear transformations and volumes

Invertibility and volume

When is a 2×2 matrix invertible?

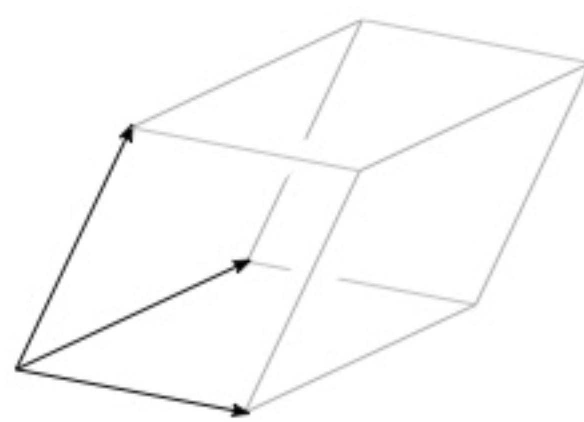
When the rows (or columns) don't lie on a line $\Leftrightarrow$ the corresponding parallelogram has non-zero area



$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

When is a 3×3 matrix invertible?

When the rows (or columns) don't lie on a plane $\Leftrightarrow$ the corresponding parallelepiped (3D parallelogram) has non-zero volume



Same for $n \times n$!

The definition of determinant

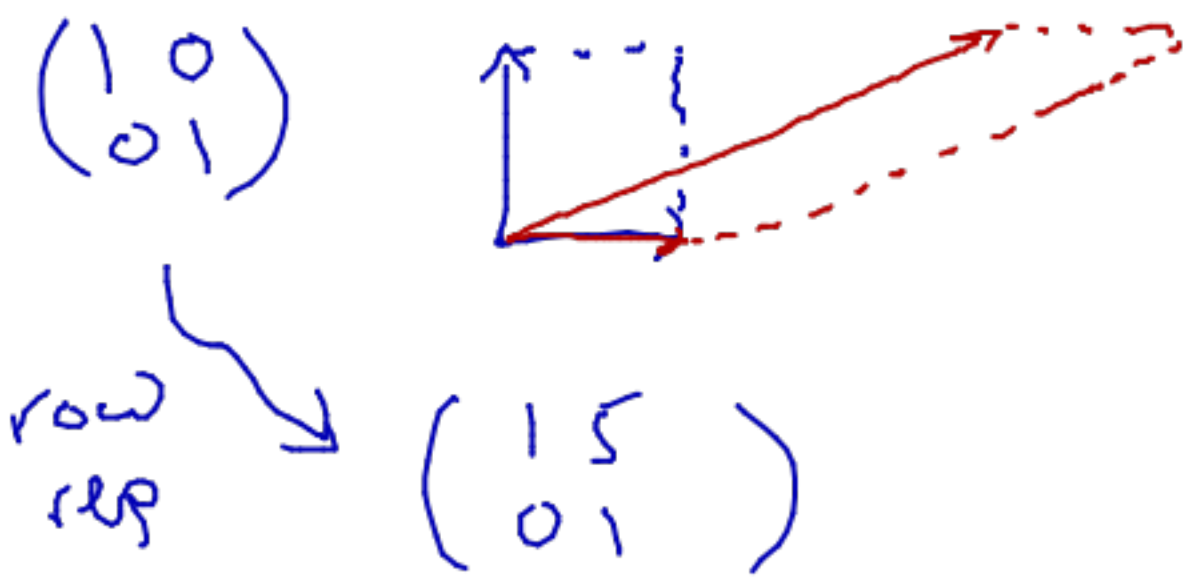
The determinant of a *square* matrix is a number so that

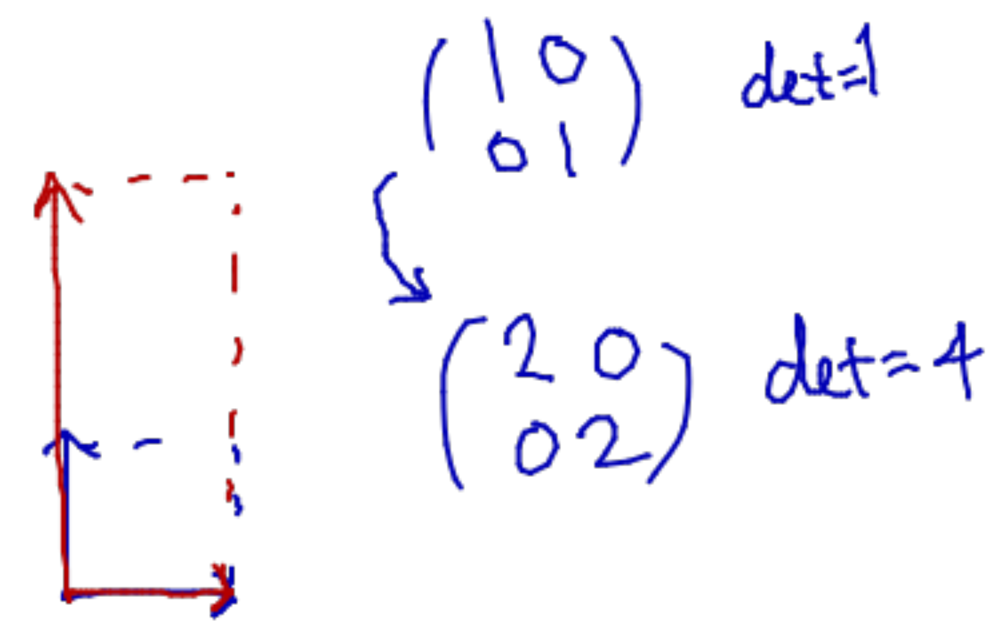
1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

Later in Sec 4.2: Formula for det

Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.





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$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 34 \qquad \det = 1 \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{\text{2 row swaps}} I_3$$

Problem. Just using these rules, compute the determinants:

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

det = 1 (row rep → $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, det = 1)

det = −1 (1 row swap → I_3, det = 1)

row scale: det = 17

det = 24: scale by 1/4 → $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/4 \\ 0 & 0 & 6 \end{pmatrix}$ det = 6; scale 1/6 → $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/4 \\ 0 & 0 & 1 \end{pmatrix}$ det = 1 $\xrightarrow{\text{row rep}} I_3$

A basic fact about determinants

Fact. If A has a zero row, then $\det(A) = 0$.

Why does this follow from the definition?

$$7 \cdot \det\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 0 & 0 & 0\end{pmatrix} = \det\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7\cdot 0 & 7\cdot 0 & 7\cdot 0\end{pmatrix}$$

A

$$\longrightarrow \det A = 0$$

A first formula for the determinant

Fact. Suppose we row reduce A. Then

$$\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

example $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 7 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{scale } 1/2} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 7/2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{rep}]{\text{row}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$

$\det A = (-1)^0 \dfrac{1 \cdot 1 \cdot 1}{1/2} = 2$

What is the determinant of a matrix in row echelon form?

$\det A$ = product of diag entries

Use the fact to get a formula for the determinant of any 2×2 matrix.

Consequence of the above fact:

Fact. $\det A \neq 0 \Leftrightarrow A$ invertible

Computing determinants

...using the definition in terms of row operations

$$\det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} =$$

A Mathematical Conundrum

We have this definition of a determinant, and it gives us a way to compute it.

But: we don't know that such a determinant function exists.

More specifically, we haven't ruled out the possibility that two different row reductions might gives us two different answers for the determinant.

Don't worry! It is all okay.

We already gave the key idea: that determinant is just the volume of the corresponding parallelepiped. You can read the proof in the book if you want.

Fact 1. There is such a number $\det$ and it is unique.

Properties of the determinant

Fact 1. There is such a number $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$ ok, now we need to say what transpose is

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

If you want the proofs, see the book. Actually Fact 1 is the hardest!

Powers

Fact 5. $\det(AB) = \det(A)\det(B)$

Use this fact to compute

$$\det\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix}^5\right)$$

What is $\det(A^{-1})$?

Powers

Poll

Suppose we know A^5 is invertible. Is A invertible?

1. yes
2. no
3. maybe

Areas of triangles

What is the area of the triangle in $\mathbb{R}^2$ with vertices $(1, 2)$, $(4, 3)$, and $(2, 5)$?

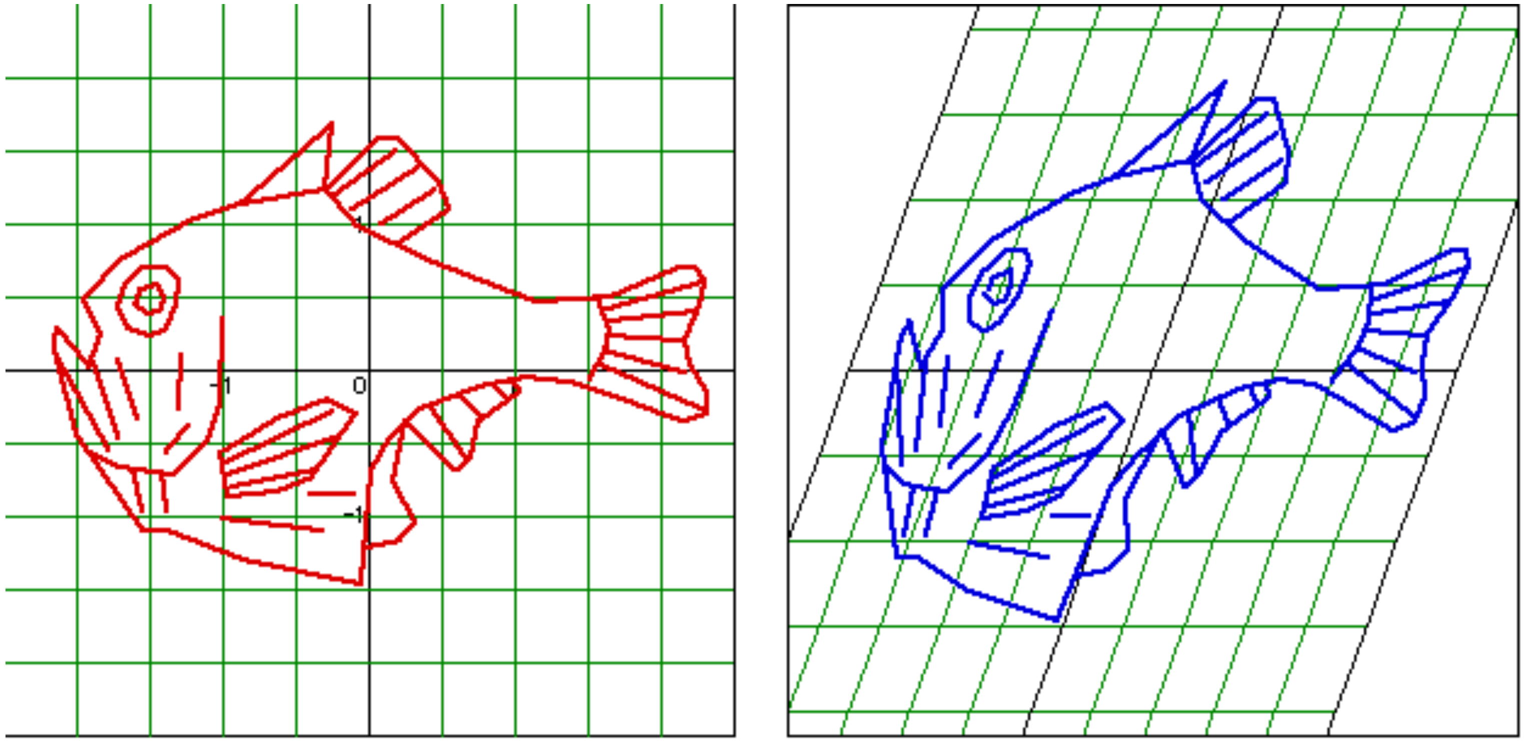
What is the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1, 2)$, $(4, 3)$, $(2, 5)$, and $(5, 6)$?

Determinants and linear transformations

Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

Summary of Sections 4.1 and 4.3

Say $\det$ is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\dfrac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

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Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S)$.

Section 4.2

Cofactor expansions

A formula for the determinant

We will give a recursive formula.

First some terminology:

$A_{ij} = ij$th minor of A
$A_{ij} = (n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column

$C_{ij} = (-1)^{i+j} \det(A_{ij})$
$= ij$th cofactor of A

Finally:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Or:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

Determinants

Compute

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5\left(+\det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}\right) + 1\left(-\det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix}\right)$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad + 0 \cdot (\ ?\ )$$

$$= 5\cdot(-3) + 1\cdot(7)$$

$$= -15 + 7$$

$$= -8$$

A formula for the determinant

Another formula for 3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Use this formula to compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Expanding across other rows and columns

The formula we gave for $\det(A)$ is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in} \text{ for any fixed } i$$
$$\det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj} \text{ for any fixed } j$$

Or:

$$\det(A) = a_{i1}(\det(A_{i1})) - a_{i2}(\det(A_{i2})) + \cdots \pm a_{in}(\det(A_{in}))$$
$$\det(A) = a_{1j}(\det(A_{1j})) - a_{2j}(\det(A_{2j})) + \cdots \pm a_{nj}(\det(A_{nj}))$$

Compute:

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

Determinants of triangular matrices

If A is upper (or lower) triangular, $\det(A)$ is easy to compute:

$$\det \begin{pmatrix} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Determinants

Poll

What is the determinant?

$$\det \begin{pmatrix} 0 & 7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

A formula for the inverse
(from Section 3.3)

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$n \times n$ matrices

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{pmatrix} \\ &= \frac{1}{\det(A)} (C_{ij})^T \end{aligned}$$

Check that these agree!

The proof uses Cramer's rule (see the notes on the course home page. We're not testing on this - it's just for your information.)

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.