Announcements Mar 30

- Class participation (Piazza polls) is optional for the rest of the semester.
- We will use Blue Jeans Meetings for the rest of the semester.
- The new schedule is on the web page.
- Midterm 3 on April 17
- WeBWorK 5.1 due Thu April 2.
- Official quiz on Friday on Canvas. It will be open all day Friday, but there will be a time limit.
- My office hours Monday 3-4 and Wed 2-3 on Blue Jeans
- TA office hours on Blue Jeans (you can go to any of these!)
  - Isabella Mon 11-12, Wed 11-12
  - Kyle Wed 3-5, Thu 1-3
  - Kalen Mon/Wed 1-2
  - Sidhanth Tue 10-12
- Supplemental problems and practice exams on the master web site

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \qquad \text{or}$$
> $$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), principal component analysis, Google, Netflix, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

Chapter 5

Eigenvectors and eigenvalues

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

eigen $=$ *characteristic*

So Av points in the same direction as v.

This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

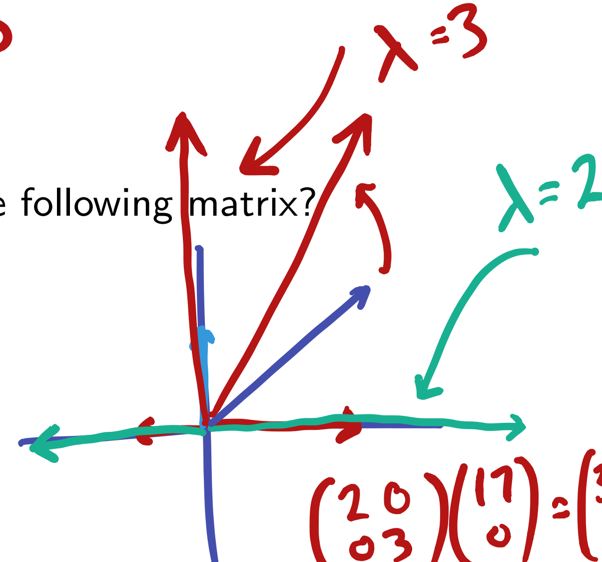
$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

2, 3

Can you find any eigenvectors, eigenvalues for the following matrix?

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$



$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 10 \\ 21 \end{pmatrix}$$

not eigenvector

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 17 \\ 0 \end{pmatrix} = \begin{pmatrix} 34 \\ 0 \end{pmatrix}$$

What happens when you apply larger and larger powers of A to a vector?

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix}, \quad \lambda = 2$$

Check $Av = 2v$ $\quad A\begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 64 \\ 16 \\ 4 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4$$

$$\begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \lambda = 4$$

How do you check?

$$\begin{pmatrix} -2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 12 \\ -3 \end{pmatrix}$$

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

Want a v ($\neq 0$) so that $Av = 3v$

$\iff Av - 3v = 0$

$\iff Av - 3Iv = 0$

$\iff \underbrace{(A - 3I)}_{\text{matrix}} v = 0$

$\iff v$ in $\mathrm{Nul}(A - 3I)$

$A - 3I = \boxed{\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}}$

not invertible!

So 3 is an eigenvalue

eigenvector $\boxed{\begin{pmatrix} 4 \\ -1 \end{pmatrix}}$

Happens when $A - 3I$ not invertible.

$\iff \boxed{\det A - 3I = 0}$

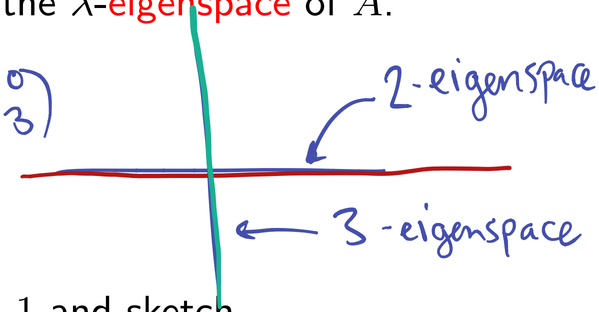
What is a general procedure for finding eigenvalues?

Eigenspaces

example $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: eigenvalue 2, eigenspace $\mathbb{R}^2$

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$



Why is this a subspace?

$$\boxed{\textbf{Fact.}\ \lambda\text{-eigenspace for } A = \mathrm{Nul}(A - \lambda I)}$$

Example. Find the eigenspaces for $\lambda = 2$ and $\lambda = -1$ and sketch.

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \qquad A - 2I = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\underline{\lambda = 2}$

$$A - 2I = \begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix}$$

subtract 2 from diag $\begin{pmatrix} \cdot & \\ & \cdot \end{pmatrix}$

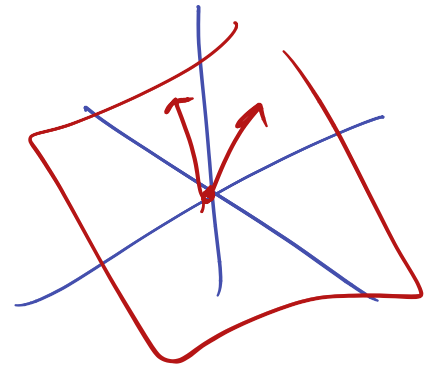
$$\boxed{\lambda = -1}$$

$$\xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \xrightarrow[\text{param form}]{\text{vector}} \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

one eigenv.

The 2-eigenspace is the line of slope 2 thru origin.

Find a basis for the 2–eigenspace:



$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

Subtract 2 from diag:

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{param form}]{\text{vect.}}$$ 2 vectors:

2-eigensp. is a plane in $\mathbb{R}^3$

Eigenvalues

And invertibility

Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A

Why?

Already said:

0 $\cancel{\lambda}$ eigenval of A $\Longleftrightarrow$ $A \cancel{- \lambda I}$ not inv.

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

$$A = \begin{pmatrix} 5 & 7 & 9 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 7 & 3 & 0 \\ 9 & 2 & 4 \end{pmatrix}$$

Check: 5 is an eigenval:

$$A - 5I = \begin{pmatrix} 0 & 7 & 9 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

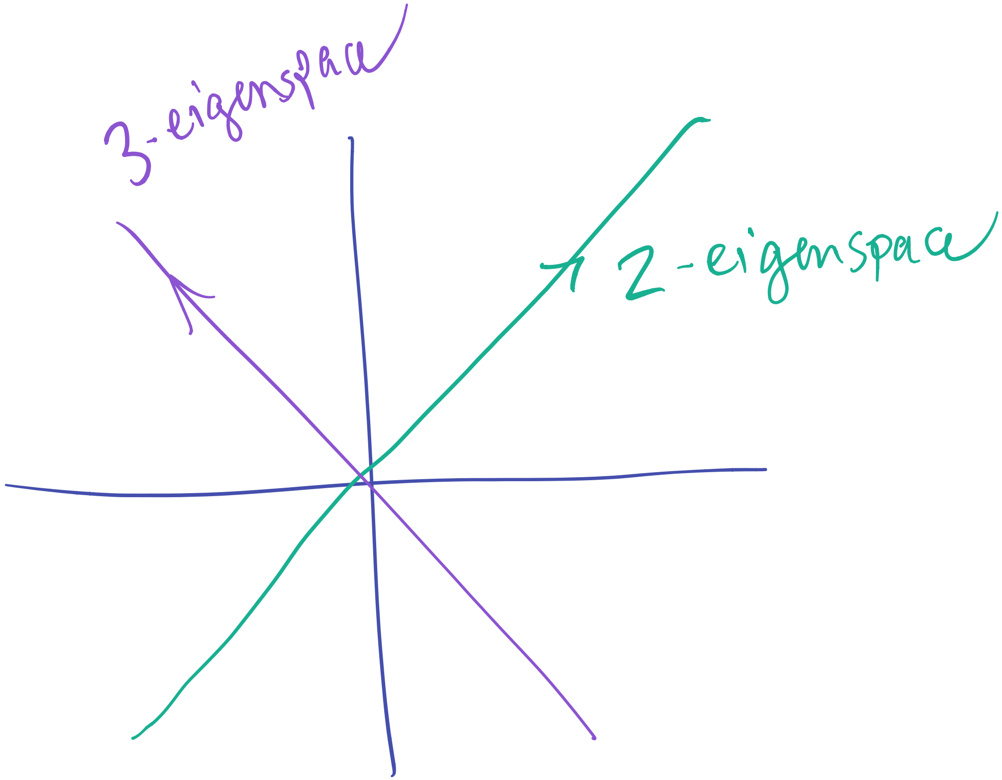
<u>not</u> inv.

eigenvals: 5, 3, 4 in both cases

Eigenvalues
Distinct eigenvalues

Fact. If $v_1 \dots v_k$ are ~~distinct~~ eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.

Why?

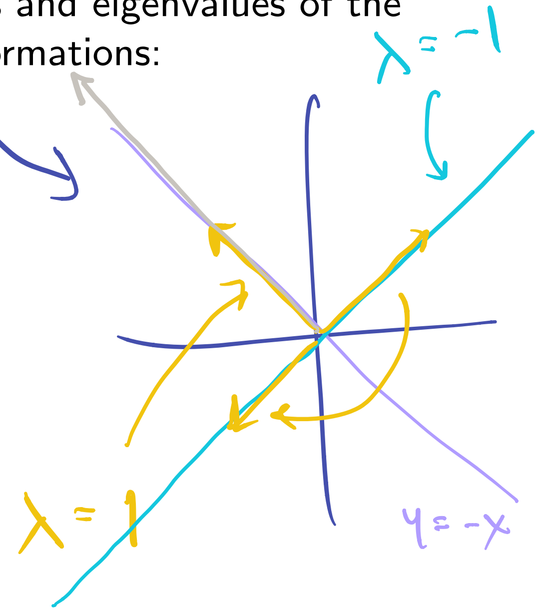


Eigenvalues geometrically

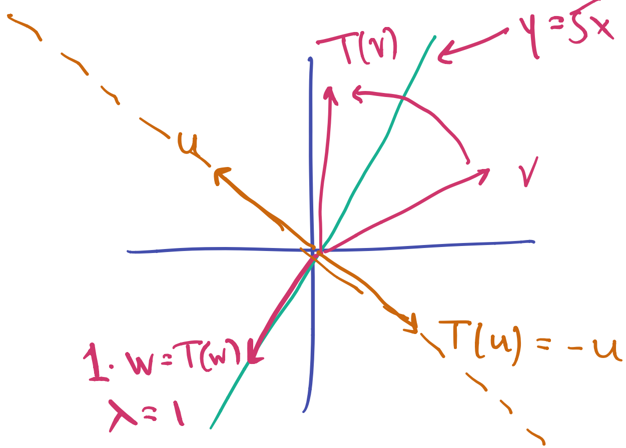
If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

- Reflection about the line $y = -x$ in $\mathbb{R}^2$
- Orthogonal projection onto the x-axis in $\mathbb{R}^2$
- Rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)
- Scaling of $\mathbb{R}^2$ by 3
- (Standard) shear of $\mathbb{R}^2$
- Orthogonal projection to the xy-plane in $\mathbb{R}^3$



Demo



Review for Section 5.1

True or false: The zero vector is an eigenvector for every matrix.

What are the eigenvalues for a reflection about a line in $\mathbb{R}^2$?

How many different eigenvalues can there be for an $n \times n$ matrix?

Section 5.2

The characteristic polynomial

Characteristic polynomial

Recall:

$$\lambda \text{ is an eigenvalue of } A \iff A - \lambda I \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial, the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

The eigenrecipe

Say you are given an square matrix A.

Step 1. Find the eigenvalues of A by solving

$$\det(A - \lambda I) = 0$$

Step 2. For each eigenvalue λ_i the λ_i-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

To find a basis, find the vector parametric solution, as usual.

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial of an $n \times n$ matrix A is a polynomial with leading term $(-1)^n$, next term $(-1)^{n-1}\mathrm{trace}(A)$, and constant term $\det(A)$:

$$(-1)^n \lambda^n + (-1)^{n-1}\mathrm{trace}(A)\lambda^{n-1} + \cdots + \det(A)$$

So for a 2×2 matrix:

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A)$$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}$$

What are the eigenvalues? Hint: Don't multiply everything out!

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Without the hint, could use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

Review of Section 5.2

True or false: every $n \times n$ matrix has an eigenvalue.

True or false: every $n \times n$ matrix has n distinct eigenvalues.

True or false: the nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.

What are the eigenvalues for the standard matrix for a reflection?