Announcements Mar 9

- Midterm 3 on April 10
- WeBWorK on Chapter 4 due Thursday
- My office hours Monday 3-4 and Wed 2-3 in Skiles 234
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- PLUS sessions Mon/Wed 6-7 LLC West with Miguel
- Supplemental problems and practice exams on the master web site

Midsemester Questionnaire

Lecture

You: The lecture has moved at a good pace with usually enough time to write everything on the slides down in my notes. The TV screens on the side of the class are extremely helpful when sitting in the first couple rows; however facing them more towards us would help us see them better with less glare from the lighting.

Lecture

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Me: Please remind me!

Lecture

You: I think more questions related to our tests/quizzes should be reviewed at lecture instead of just going over slides I could do in my room.

You: Lecture is very good, concepts are well taught, great professor. I think it would be more helpful if we could cover more examples in class.

Lecture

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You: Lecture is very good, concepts are well taught, great professor. I think it would be more helpful if we could cover more examples in class.

Me: Will try to add more questions to the end of each section.

Lecture

You: Lecture is good but more structured lecture slides would be great . If you post your notes that would be WONDERFUL.

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Me: They are posted!

WeBWorK

You: Webwork is fine, I just wish we had unlimited attempts for all of them. I also think it would be helpful if it gave us a step by step example for each problem because sometimes I do not even know how to start the problems.

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Me: It's better this way!

WeBWorK

You: The homework has not been helpful for me at all. Sometimes the true or false questions are helpful in understanding concepts, but the numbers are typically too large and require the use of a calculator. I would love it if the homework mirrored what the actual test and quizzes asked, instead of throwing large numbers and complex questions at us.

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Me: The goal is not just to prepare for exams...

Studio

You: The TAs are fantastic! They are super helpful in studio and I love how Isabella gives a little recap on the lectures before we start on the worksheet. I would also like if the worksheet solutions and explanations were posted on Canvas. (sorry if they already are, I have looked for them and cannot find)

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Me: See the master course web site!

Preparation

You: I feel like I have to spend a great amount of time studying/figuring/visualizing things out on my own to be able to get it. Recently I have felt more prepared but only because I invested more time than I had wanted on figuring stuff out.

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Me: It might take some time!

Preparation

You: I think I know how to succeed in this class, I just have so much else going on in my other classes/life that whenever I do have free time I find it difficult to want to spend that time going to office hours/tutoring/reading the textbook. That is completely on me though. I think I am going to try to come to office hours more often and get a private tutor because that is what I need to do so I can reach my goal of getting at least a B.

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Me: I know!

Preparation

You: I feel like succeeding in this class is hard if you aren't pre-exposed to it. It's so different from all math classes we've taken so far, and it's tough to teach it as well. I've tried tutoring and study groups, but it depends on who your tutor is and who you're studying with. Overall, the class is honestly really hard for most people (from what I hear) and I feel like there should be more help offered for it specifically.

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Me: Yes, it is hard!

Professor

You: Prof Margalit is a king!!! He is super nice, patient, incredibly smart, and good at explaining everything. I would love if he could do more test example problems in lecture though. Thank you for being great, I am just terrible at this kind of math.

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Me: You are a king!

Professor

You: keeps lectures interesting and very flexible with office hours. explaining notation a little more clearly would be very helpful because a lot of the time the only thing between me and understanding is trying to read the notation

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You: keeps lectures interesting and very flexible with office hours. explaining notation a little more clearly would be very helpful because a lot of the time the only thing between me and understanding is trying to read the notation

Me: Please ask!

Other

You: PLUS sessions save lives

Other

You: PLUS sessions save lives

Me: Thanks Miguel!

Overall

Is this an effective instructor?

You: 4.56 / 5

Overall

Is this an effective instructor?

You: 4.56 / 5

Me: Will try harder!

Where are we?

- We have studied the problem $Ax = b$
- We next want to study $Ax = \lambda x$
- At the end of the course we want to almost solve $Ax = b$

We need determinants for the second item.

Section 4.1

The definition of the determinant

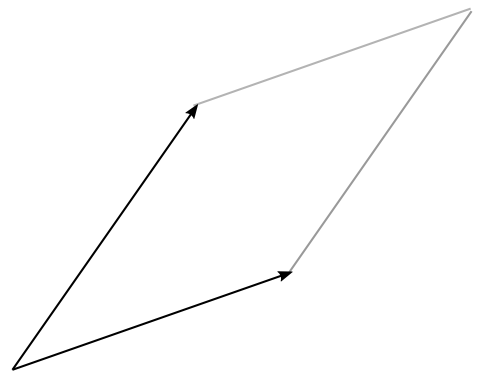
Outline of Sections 4.1 and 4.3

- Volume and invertibility
- A definition of determinant in terms of row operations
- Using the definition of determinant to compute the determinant
- Determinants of products: $\det(AB)$
- Determinants and linear transformations and volumes

Invertibility and volume

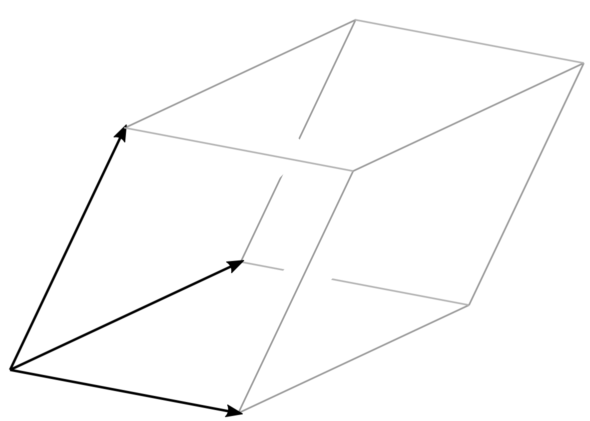
When is a 2×2 matrix invertible?

When the rows (or columns) don't lie on a line $\Leftrightarrow$ the corresponding parallelogram has non-zero area



When is a 3×3 matrix invertible?

When the rows (or columns) don't lie on a plane $\Leftrightarrow$ the corresponding parallelepiped (3D parallelogram) has non-zero volume



Same for $n \times n$!

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.

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Problem. Just using these rules, compute the determinants:

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

A basic fact about determinants

Fact. If A has a zero row, then $\det(A) = 0$.

Fact. If A is in row echelon form then $\det(A)$ is the product of the diagonal entries.

Why do these follow from the definition?

A first formula for the determinant

Fact. Suppose we row reduce A. Then

$$\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$$

Use the fact to get a formula for the determinant of any 2×2 matrix.

Consequence of the above fact:

Fact. $\det A \neq 0 \Leftrightarrow A$ invertible

Computing determinants

...using the definition in terms of row operations

$$\det\begin{pmatrix}0&1&0\\1&0&1\\5&7&-4\end{pmatrix} =$$

det=9

det = −9

$$\rightsquigarrow \begin{pmatrix}1&0&1\\0&1&0\\5&7&-4\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&0&1\\0&1&0\\0&7&-9\end{pmatrix} \quad \det = -9$$

$$\rightsquigarrow \begin{pmatrix}1&0&1\\0&1&0\\0&0&-9\end{pmatrix} \quad \det = -9$$

A Mathematical Conundrum

We have this definition of a determinant, and it gives us a way to compute it.

But: we don't know that such a determinant function exists.

More specifically, we haven't ruled out the possibility that two different row reductions might gives us two different answers for the determinant.

Don't worry! It is all okay.

We already gave the key idea: that determinant is just the volume of the corresponding parallelepiped. You can read the proof in the book if you want.

Fact 1. There is such a number $\det$ and it is unique.

Properties of the determinant

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Fact 1. There is such a number $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$ ok, now we need to say what transpose is

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

If you want the proofs, see the book. Actually Fact 1 is the hardest!

Powers

Fact 5. $\det(AB) = \det(A)\det(B)$

Use this fact to compute

$$\det\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix}^5\right)$$

$= 9^5$

What is $\det(A^{-1})$?

$= 9^{-1} = 1/9$

$\det A \det A^{-1} = \det I = 1$

Powers

Poll

Suppose we know A^5 is invertible. Is A invertible?

1. yes
2. no
3. maybe

$$\begin{pmatrix} 5 & * & * \\ 0 & 6 & * \\ 0 & 0 & 7 \end{pmatrix}$$

$$\det A^5 \neq 0$$

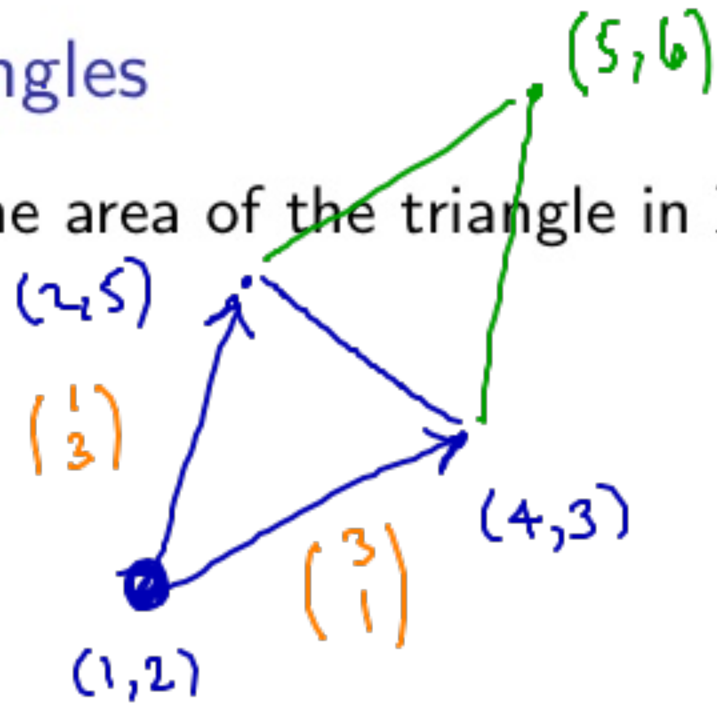
$$\Longrightarrow \det A \neq 0$$

Section 4.3

The determinant and volumes

Areas of triangles

What is the area of the triangle in $\mathbb{R}^2$ with vertices $(1,2)$, $(4,3)$, and $(2,5)$?



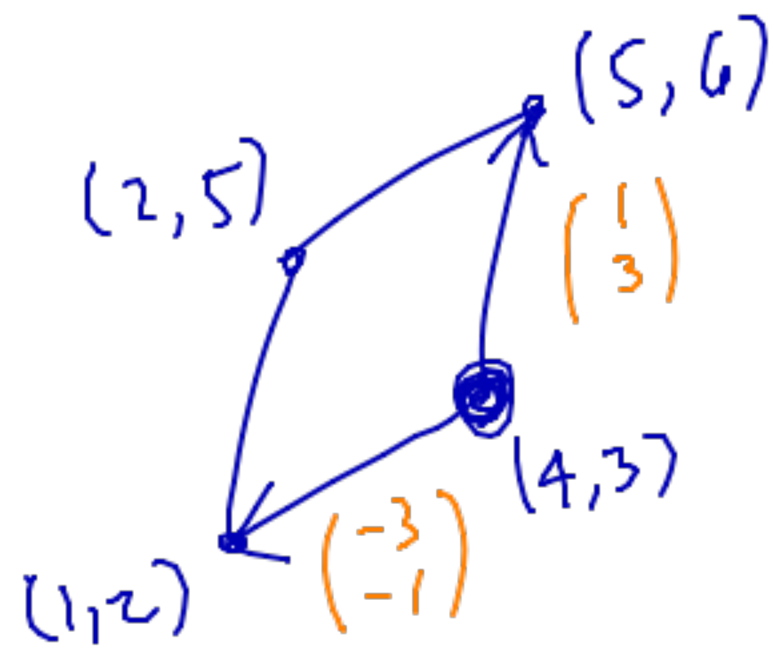
$$= \frac{1}{2}\left|\det\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\right|$$

$$= \frac{1}{2}\left|-8\right| = 4$$

What is the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1,2)$, $(4,3)$, $(2,5)$, and $(5,6)$?

$$8$$

$$\left|\det\begin{pmatrix} 1 & -3 \\ 3 & -1 \end{pmatrix}\right| = 8$$

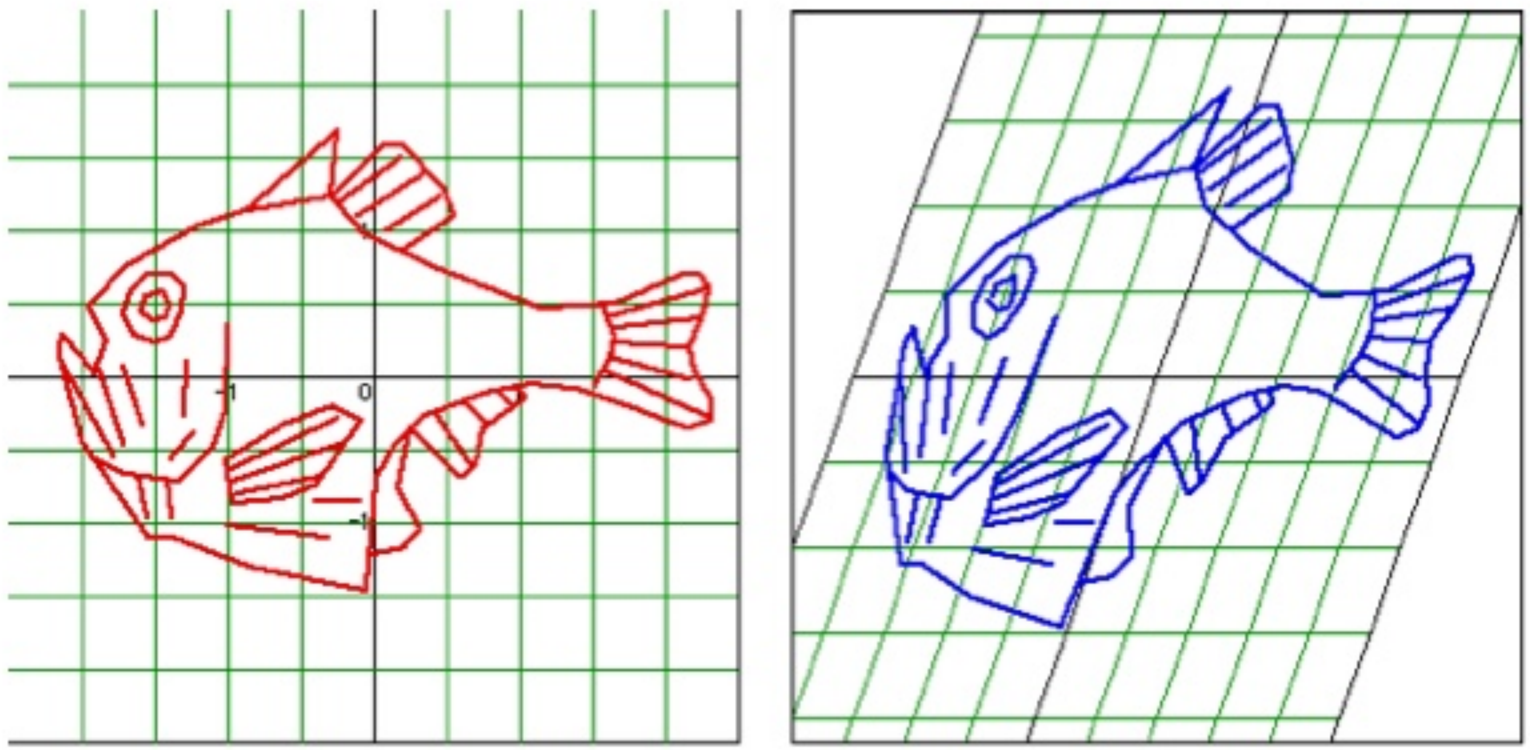


Determinants and linear transformations

Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



If $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then area(fish) $= 1 \cdot$ area(T(fish))

Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

Summary of Sections 4.1 and 4.3

Say $\det$ is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

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Fact 5. $\det(AB) = \det(A)\det(B)$ important!

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Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S)$.

Section 4.2

Cofactor expansions

Outline of Section 4.2

- We will give a recursive formula for the determinant of a square matrix.

A formula for the determinant

We will give a recursive formula.

First some terminology:

A_{ij} = ijth minor of A
$A_{ij} = (n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column

$C_{ij} = (-1)^{i+j}\det(A_{ij})$
$= ij$th cofactor of A

Finally:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Or:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

A formula for the determinant

For the recursive formula:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

Need to start somewhere...

1×1 matrices

$$\det(a_{11}) = a_{11}$$

2×2 matrices

$$\begin{aligned} \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{11}\det(A_{11}) + a_{12}(-\det(A_{12})) \\ &= a_{11}(a_{22}) + a_{12}(-a_{21}) \end{aligned}$$

A formula for the determinant

3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \cdots$$

You can write this out. And it is a good exercise. But you won't want to memorize it.

Determinants

Compute

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5 \cdot \det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 1 \det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix}$$

$$+ 0 \cdot (\text{don't care})$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (5 \cdot -3) - (1 \cdot -7)$$

$$= -15 + 7 = -8$$

$$+5 \cdot (\qquad) - 1 \cdot (\qquad)$$

$$+0 \cdot (\qquad)$$

A formula for the determinant

Another formula for 3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Use this formula to compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Expanding across other rows and columns

The formula we gave for $\det(A)$ is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in} \text{ for any fixed } i$$
$$\det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj} \text{ for any fixed } j$$

Or:

$$\det(A) = a_{i1}(\det(A_{i1})) - a_{i2}(\det(A_{i2})) + \cdots \pm a_{in}(\det(A_{in}))$$
$$\det(A) = a_{1j}(\det(A_{1j})) - a_{2j}(\det(A_{2j})) + \cdots \pm a_{nj}(\det(A_{nj}))$$

Compute:

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$0 \cdot (\quad) - 0 \cdot (\quad) + 1 \cdot \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1$$

Determinants of triangular matrices

If A is upper (or lower) triangular, $\det(A)$ is easy to compute:

$$\det\begin{pmatrix} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$2 \cdot \det\begin{pmatrix} 1 & 2 & -3 \\ 0 & 5 & 9 \\ 0 & 0 & 10 \end{pmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$2 \cdot \left(1 \cdot \det\begin{pmatrix} 5 & 9 \\ 0 & 10 \end{pmatrix} \right)$$

$$= 2 \cdot 1 \cdot 5 \cdot 10$$

Determinants

Poll

What is the determinant?

$$\det \begin{pmatrix} 0 & 7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

A formula for the inverse
(from Section 3.3)

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$n \times n$ matrices

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix} \\ &= \frac{1}{\det(A)} (C_{ij})^T \end{aligned}$$

Check that these agree!

The proof uses Cramer's rule (see the notes on the course home page. We're not testing on this - it's just for your information.)

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.