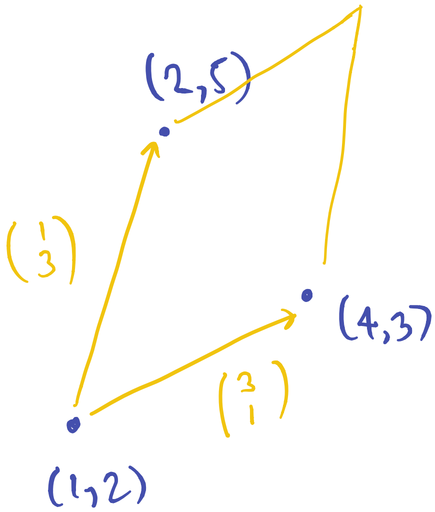


$$\frac{\left|\det\begin{pmatrix}1&3\\3&1\end{pmatrix}\right|}{2} = \frac{8}{2} = 4$$

Find eigenvals of

$$\begin{pmatrix}7&0&\overset{+3}{\cancel{4}3}\\-3&2&-3\\4&2&0\end{pmatrix}$$

$$\det\begin{pmatrix}\boxed{7-\lambda \quad 0 \quad -3}\\-3 \quad 2-\lambda \quad -3\\4 \quad 2 \quad -\lambda\end{pmatrix}$$

$-14+4\lambda$
$-6-8+4\lambda$
$-6-4(2-\lambda)$

$$= +(7-\lambda)\det\begin{pmatrix}2-\lambda&-3\\2&-\lambda\end{pmatrix}$$

$$-0\cdot ? \quad + (-3)\det\begin{pmatrix}-3&2-\lambda\\4&2\end{pmatrix}$$

$$= (7-\lambda)(\lambda^2-2\lambda+6) + 3(-14+4\lambda)$$

Technique #1. Don't multiply out

Doesn't work here – no common factor.

So multiply out:

$$7\lambda^2 - 14\lambda + 42 - \lambda^3 + 2\lambda^2 - 6\lambda + \cancel{42 - 12\lambda} \quad -42 + 12\lambda$$

$$-\lambda^3 + 9\lambda^2 - 8\lambda$$

Technique #2. Guess a root, factor

Tech 2a. No const term $\Rightarrow$ 0 a factor

not explicitly tested this semester.

Tech 2b... The problem tells you one eigenval.

Tech 2c... Rational root theorem: try $\dfrac{\text{factor of const term}}{\text{factor of leading term}}$.

In our case:

$$-\lambda(\lambda^2 - 9\lambda + 8)$$

$$-\lambda(\lambda - 8)(\lambda - 1)$$

$$\rightsquigarrow \lambda = 0, 8, 1$$

Using Tech 2b:

If $\lambda = 2$ is a root,

factor out $(\lambda - 2)$

Using Tech 2c: $+1 -3 +2$

$$-\lambda^3 + 3\lambda + 2$$

Factors of const term: $\pm 1, \pm 2$
Factors of leading term: ± 1
So only possible rational roots are:

$$\boxed{\pm 1, \pm 2}$$

Plug in 1 ✗
-1 ✓
2 ✓
-2 ✗

So: $\lambda = 2$ is an eigenvalue.

$$\begin{array}{r|l} & -\lambda^2 - 2\lambda - 1 \\ \hline \lambda - 2 & -\lambda^3 + 0\lambda^2 + 3\lambda + 2 \\ & -(-\lambda^3 + 2\lambda^2) \\ \hline & -2\lambda^2 + 3\lambda + 2 \\ & -(-2\lambda^2 + 4\lambda) \\ \hline & -\lambda + 2 \end{array}$$

$$-(\lambda - 2)(\lambda^2 + 2\lambda + 1)$$

$$\lambda = 2, \quad \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

$$\begin{pmatrix} 7 & 0 & \overset{+3}{\cancel{-3}} \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

$$\boxed{\begin{array}{l} (-1)^3 \lambda^3 + (-1)^2 \operatorname{tr}(A) \lambda^2 \\ \quad + \; ? \; \lambda \; + \det A \end{array}}$$

$$7 \det \begin{pmatrix} 2 & -3 \\ 2 & 0 \end{pmatrix} + 3 \det \begin{pmatrix} -3 & 2 \\ 4 & 2 \end{pmatrix}$$

$$= 7 \cdot 6 + 3 \cdot (-14)$$

$$= 2$$

$$= -\lambda^3 + \operatorname{tr}(A) \lambda^2 + \underline{\quad\quad} + \det A$$

$$= -\lambda^3 + 9\lambda^2 + \underline{\quad\quad} + 2$$

Diagonalization

$$\frac{1}{4} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

<u>Step 1</u> Find eigenvals

$$\det \begin{pmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{pmatrix}$$

$$\lambda^2 - 10\lambda + 16$$

$$(\lambda - 8)(\lambda - 2)$$

$$\rightarrow \lambda = 2, 8$$

Step 2 Find eigenvectors

$\underline{\lambda=2}$ $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ eigenvec: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\boxed{\begin{aligned} &\underline{\text{Fact from 5.1}} \quad v^{\neq 0} \text{ is a } \lambda\text{-eigenvec} \\ &\quad \iff v \text{ solves } (A-\lambda I)x = 0 \\ &\quad \iff v \text{ in } \mathrm{Nul}\,(A-\lambda I) \end{aligned}}$

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} ab-ba \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda=8}$ $\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$\underline{\text{or}}$ $\frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$

If you Diagonalize A as CDC^{-1}

then $kA = C(kD)C^{-1}$

Diagonalize if possible:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{pmatrix} = +(1-\lambda)\left((1-\lambda)^2-4\right)$$

$$= (1-\lambda)(\lambda^2-2\lambda-3)$$

$$= (1-\lambda)(\lambda-3)(\lambda+1)$$

$$\rightsquigarrow \lambda = 1, -1, 3$$

Yes....

What about $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$?

$$\det \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda)(1-\lambda)$$

$$\rightsquigarrow \lambda = 2, 1$$

alg mult = 2

This will be diag'able if:

2-eigenspace is... 2-dim.

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x &= -z \\ y &= y \\ z &= z \end{aligned}$$

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2-eigensp is 2-dim!

So the orig. matrix is diagonalizable

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}^{-1}$$

2-eigenv. 1-eigenv.

1-eigensp:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\boxed{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}$ eigenval 1 ← alg mult. 2

1-eigenspace: $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ dimension = 1

$\underbrace{\text{dim 1-eigenspace}}^{\text{geom. dim of 1}} <$ alg. mult. of 1

Always true that:

$1 \leq \text{dim } \lambda\text{-eigensp} \leq$ alg mult. of λ → add up to λ

For diag'ility: the $\leq$ must be $=$ for all λ.

So $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ not diag'able.

Sample Q. A is 3×3

Eigenvals 5, 7

Dim of 5-eigensp is 2

Must it be true that A is diag'able?

YES.

Example $A = 10 \times 10$ matrix (no complex eigenvalues)

premise

λ	alg. mult	dim of eigensp
3	2	≤ 2, ≥ 1
5	5	≤ 5, ≥ 1
6	3	≤ 3, ≥ 1

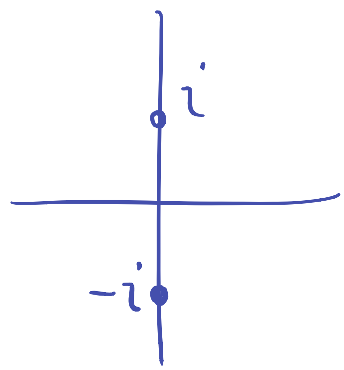
add up to 10

conclude from premise

all of these must be = for diag'ability.

Example from class

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow \lambda = \pm i$$



i-eigenspace: $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \longrightarrow \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 \\ +i \end{pmatrix}$

$\boxed{-i\text{-eigenspace: } \begin{pmatrix} -1 \\ -i \end{pmatrix}}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ i \end{pmatrix} = i \cdot \begin{pmatrix} -1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ i^2 \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix}$$

Check $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix}$

Fact. ① Real eigenvalues: planes stretched.
② Complex eigenvals: planes rotated and stretched.

Example. $A = 3\times 3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

eigenvalues: $i, -i, 2$