Announcements Mar 4

- Midterm 2 on Friday
- WeBWorK 3.4, 3.5, and 3.6 due Thursday
- TA office hours in Skiles 230 (you can go to any of these!)
  - Isabella Thu 2-3
  - Kyle Thu 1-3
  - Kalen Mon/Wed 1-1:50
  - Sidhanth Tue 10:45-11:45
- Review sessions
  - Kalen 7 pm Thu online
  - Sidhanth 7 pm tonight
- PLUS session with Miguel tonight 5-7
- Supplemental problems and practice exams on the master web site
Review for Midterm 2
Section 2.6 Summary

- A **subspace** of $\mathbb{R}^n$ is a subset $V$ with:
  1. The zero vector is in $V$.
  2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$.
  3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu \in V$.

- Two important subspaces: $\text{Nul}(A)$ and $\text{Col}(A)$

- Find a spanning set for $\text{Nul}(A)$ by solving $Ax = 0$ in vector parametric form

- Find a spanning set for $\text{Col}(A)$ by taking pivot columns of $A$ (not reduced $A$)

- Four things are the same: subspaces, spans, planes through 0, null spaces

Let $V$ be the subset of $\mathbb{R}^3$ consisting of the $x$-axis, the $y$-axis, and the $z$-axis. Which properties of a subspace does $V$ fail?

Find a spanning set for the plane in $\mathbb{R}^3$ defined by $x + y - 2z = 0$. 
Section 2.7 Summary

- A basis for a subspace \( V \) is a set of vectors \( \{v_1, v_2, \ldots, v_k\} \) such that
  1. \( V = \text{Span}\{v_1, \ldots, v_k\} \)
  2. \( v_1, \ldots, v_k \) are linearly independent

- The number of vectors in a basis for a subspace is the dimension.

- Find a basis for \( \text{Nul}(A) \) by solving \( Ax = 0 \) in vector parametric form

- Find a basis for \( \text{Col}(A) \) by taking pivot columns of \( A \) (not reduced \( A \))

- Basis Theorem. Suppose \( V \) is a \( k \)-dimensional subspace of \( \mathbb{R}^n \). Then
  - Any \( k \) linearly independent vectors in \( V \) form a basis for \( V \).
  - Any \( k \) vectors in \( V \) that span \( V \) form a basis.

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Find a basis \( \{u, v, w\} \) for \( \mathbb{R}^3 \) where no vector has a zero entry.
Section 2.9 Summary

- Rank-Nullity Theorem. \( \text{rank}(A) + \dim \text{Nul}(A) = \#\text{cols}(A) \)

Let \( A \) be an \( 4 \times 6 \) nonzero matrix and suppose the columns of \( A \) are all the same. What is \( \dim \text{Nul}(A) \)?
Section 3.1 Summary

- If $A$ is an $m \times n$ matrix, then the associated matrix transformation $T$ is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\text{Col}(A)$.

- If $A$ is $n \times n$ then $T$ does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

Find a matrix $A$ so that the range of the matrix transformation $T(v) = Av$ is the line $y = 2x$ in $\mathbb{R}^2$. 
Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each $b$ in $\mathbb{R}^m$ is the output for at most one $v$ in $\mathbb{R}^n$.

- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix $A$. Then the following are all equivalent:
  - $T$ is one-to-one
  - the columns of $A$ are linearly independent
  - $Ax = 0$ has only the trivial solution
  - $A$ has a pivot
  - the range has dimension $n$

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of $T$ equals the codomain $\mathbb{R}^m$, that is, each $b$ in $\mathbb{R}^m$ is the output for at least one input $v$ in $\mathbb{R}^n$.

- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix $A$. Then the following are all equivalent:
  - $T$ is onto
  - the columns of $A$ span $\mathbb{R}^m$
  - $A$ has a pivot
  - $Ax = b$ is consistent
  - the range of $T$ has dimension $m$

Let $A$ be an $5 \times 5$ matrix. Suppose that $\dim \text{Nul}(A) = 0$. Must it be true that $Ax = e_1$ is consistent?
A function \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is linear if
\[
\begin{align*}
\ & T(u + v) = T(u) + T(v) \quad \text{for all } u, v \in \mathbb{R}^n. \\
\ & T(cv) = cT(v) \quad \text{for all } v \in \mathbb{R}^n \text{ and } c \in \mathbb{R}.
\end{align*}
\]

**Theorem.** Every linear transformation is a matrix transformation (and vice versa).

The standard matrix for a linear transformation has its \( i \)th column equal to \( T(e_i) \).

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Find the standard matrix for the linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that reflects over the line \( y = -x \) and then rotates counterclockwise by \( \pi/2 \).
Summary of Section 3.4

- Composition: \((T \circ U)(v) = T(U(v))\) (do \(U\) then \(T\))
- Matrix multiplication: \((AB)_{ij} = r_i \cdot b_j\)
- Matrix multiplication: the \(i\)th column of \(AB\) is \(A(b_i)\)
- The standard matrix for a composition of linear transformations is the product of the standard matrices.
- **Warning!**
  - \(AB\) is not always equal to \(BA\)
  - \(AB = AC\) does not mean that \(B = C\)
  - \(AB = 0\) does not mean that \(A\) or \(B\) is 0

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Find a \(2 \times 2\) matrix \(A\), not equal to \(I\), with \(A^4 = I\).
Summary of Section 3.5

- $A$ is invertible if there is a matrix $B$ (called the inverse) with
  \[ AB = BA = I_n \]

- For a $2 \times 2$ matrix $A$ we have that $A$ is invertible exactly when $\det(A) \neq 0$ and in this case
  \[ A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \]

- If $A$ is invertible, then $Ax = b$ has exactly one solution: $x = A^{-1}b$.
- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A | I_n)$.
- Invertible linear transformations correspond to invertible matrices.

Find the inverse of the matrix
\[
\begin{pmatrix}
1 & 0 & h \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Summary of Section 3.6

- Say $A = n \times n$ matrix and $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.
  1. $A$ is invertible
  2. $T$ is invertible
  3. The reduced row echelon form of $A$ is $I_n$
  4. etc.

In all questions, suppose that $A$ is an $n \times n$ matrix and that $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the associated linear transformation.

1. Suppose that the reduced row echelon form of $A$ does not have any zero rows. Must it be true that $Ax = b$ is consistent for all $b$ in $\mathbb{R}^n$?
   - YES
   - NO

2. Suppose that $T$ is one-to-one. Is it possible that the columns of $A$ add up to zero?
   - YES
   - NO

3. Suppose that $Ax = e_1$ is not consistent. Is it possible that $T$ is onto?
   - YES
   - NO
Important terms

- subspace
- column space
- null space
- basis
- dimension
- one-to-one
- onto
- linear transformation
- inverse
- Rank theorem
Good luck!

It is not the mountain we conquer but ourselves.

– Edmund Hillary