MATH 2602

LINEAR AND DISCRETE MATHEMATICS

PROF. MARGALIT

WHAT IS DISCRETE MATH?

dis·crete [dih-**skreet**] Show IPA

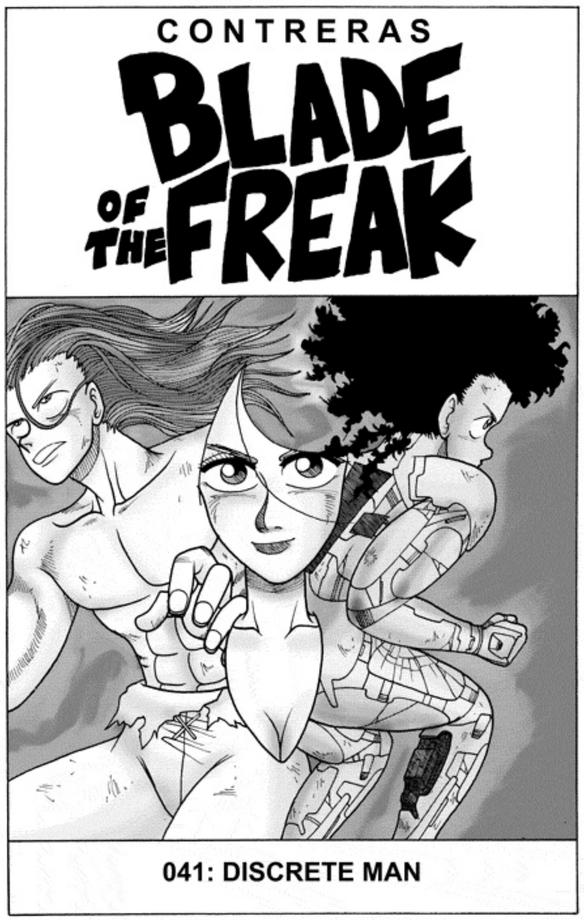
adjective

1. apart or detached from others; separate; distinct: *six discrete parts.*
2. consisting of or characterized by distinct or individual parts; discontinuous.
3. *Mathematics* .
   a. (of a topology or topological space) having the property that every subset is an open set.
   b. defined only for an isolated set of points: *a discrete variable.*
   c. using only arithmetic and algebra; not involving calculus: *discrete methods.*

dictionary.com

Discrete is the opposite of continuous.

WHAT IS DISCRETE MATH?



WHAT IS DISCRETE MATH?

CONTINUOUS	DISCRETE
real numbers	integers
measuring	counting
ideal shapes	computer images
wave	particle
differential eqn	recurrence reln.
calculus	probability graph theory algorithms

Chapter 0

Yes, there are proofs!

Knights and Knaves

Everyone is either a knight (truth-teller) or knave (liar).

1. Anna says Elsa is a knight.
   Else says she is a knight.
   What can you conclude?

2. Anna says at least one of us is a knave.
   What can you conclude?

0.1 Compound Statements

STATEMENTS

A mathematical statement is a declarative sentence that is either true or false.

Examples. 1 is a prime number.
π is a rational number.
If $1+1=3$ then $5=7$

Non-examples. What is my name?
Solve for x: $2x=10$.
Meep meep.
Et cetera.

We sometimes represent a statement by a leller.

THIS IS FALSE

Consider the following sentence:

This statement is false.

Is this a mathematical statement? Is it true or false?

New Statements from Old

And. $p \wedge q$ is true if both are.

Or. $p \vee q$ is true if at least one is.

Caution! Or has different uses
in English: can have soup or salad
need a license or passport

Not. $\neg p$ is true if p isn't.

"it is not the case that p"

IMPLICATIONS

IMPLICATION. $p \to q$ is true unless p true, q false.

"if p then q"

think about campaign promises

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

EXAMPLES. If $1=2$, I am the pope.
If I have 4 quarters, I have a dollar.

IMPLICATIONS

CONVERSE. The converse of $p \to q$ is

$q \to p$

The converse of a true statement can be true or false.

CONTRAPOSITIVE. The contrapositive of $p \to q$ is

$\neg q \to \neg p$

The contrapositive is <u>equivalent</u> to the original statement!

If you won you got a medal.
If you didn't get a medal, you didn't win.

NEGATION. $\neg(p \to q) \equiv p \land \neg q$

DOUBLE IMPLICATION. $(p \to q) \land (q \to p)$ is written

$p \leftrightarrow q$

You got a medal if and only if you won.

QUANTIFIERS

First, a propositional function is a statement with a variable that becomes a mathematical statement when a value is given to the variable:

n is even

We can also turn a propositional function into a mathematical statement using quantifiers.

For all. $\forall n \in \mathbb{Z}$ (n is even)
There exists. $\exists n \in \mathbb{Z}$ (n is even)

and combinations: $\forall n \in \mathbb{Z}\ \exists m \in \mathbb{Z}$ ($n+m$ is even)
$\exists m \in \mathbb{Z}\ \forall n \in \mathbb{Z}$ ($n+m$ is even) etc.

NEGATION. $\neg(\exists m \forall n$ ($n+m$ is even)$) \equiv$
$\forall m \exists n$ ($n+m$ is odd)

THE SECRET $\forall$

When we say:

If n is even, then $n+1$ is odd.

We really mean:

$$\forall n \ (n \text{ even} \rightarrow n+1 \text{ odd})$$

So for instance the negation is:

$$\exists n \ \neg(n \text{ even} \rightarrow n+1 \text{ odd})$$
$$\exists n \ (n \text{ even} \wedge n+1 \text{ even})$$

NEGATION

Which of the following statements are true?

(i) If $f:\mathbb{R}\to\mathbb{R}$ is an even function and $g:\mathbb{R}\to\mathbb{R}$ is an odd function then $f+g$ is an odd function.

(ii) If $f:\mathbb{R}\to\mathbb{R}$ is an even function and $g:\mathbb{R}\to\mathbb{R}$ is an odd function then fg is an odd function.

(iii) $\exists x\in\mathbb{R}\ (x^2<0)$

(iv) $\exists x\in\mathbb{R}\ \exists y\in\mathbb{R}\ (3x-2y=1 \wedge x+2y=3)$

(v) $\exists x\in\mathbb{R}\ \exists y\in\mathbb{R}\ (x+y=0 \wedge x+y=1)$

(vi) $\exists N\in\mathbb{Z}\ \forall m\in\mathbb{Z}\ (m\leq N)$

(vii) $\forall x\in\mathbb{R}\ \forall y\in\mathbb{R}\ ((x\geq 0 \wedge y\geq 0)\to xy\geq 0)$

Write the negation of each false statement.

Direct Proofs

Prove each of the following propositions.

1. For all $m \in \mathbb{Z}$, m^2+m is even.

2. If $x \geq 10$ then $x^4 \geq 100x$

3. The product of two odd functions is even.

Proofs by Cases

1. If $4 \leq n \leq 13$, then n is the sum of two primes.

2. For all $n \in \mathbb{Z}$, $n^2 - n \geq 0$.

3. It is possible to pay any (integer) number of dollars at least 6 with \$3 and \$4 bills.

Proofs by Contradiction and Contrapositive

Prove each of the following propositions.

1. The square root of an irrational number is irrational.

2. If 6 people need to eat 50 skittles, then someone must eat more than 8 skittles.

3. The function $\sqrt{x}$ is not a rational function.

$\sqrt{4}$ IS IRRATIONAL

Prop. $\sqrt{4}$ is irrational.

Proof. Suppose, for contradiction that $\sqrt{4} = p/q$, in lowest terms.

Then $4 = p^2/q^2$
so $4q^2 = p^2$
so p is even
so p^2 is divis. by 4.
so $p^2 = 4r^2$
so $4q^2 = 4r^2$
so $q^2 = r^2$
so $q = r$
so $p/q = 2r/r$ is not in lowest terms.

This is a contradiction.

More Proofs

Prove or disprove each of the following propositions.

1. No two consecutive integers are prime.

2. $\sqrt{3}$ is irrational.

3. For all $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$.

4. It is possible to tile a chessboard with dominos after two opposite corners have been removed.

TRUTH TABLES

Is the following proposition a tautology, a contradiction, or neither?

$$(\neg p \land q) \land (p \lor \neg q)$$

Can you verify your answer without truth tables?

DISJUNCTIVE NORMAL FORM

p	q	r	S
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	T

Can you find some statement S with this truth table?

Hint: disjunctive normal form

Can you find a short statement S with this truth table?

Hint: use →

DISJUNCTIVE NORMAL FORM

Is disjunctive normal form unique? In other words, is it possible to find different disjunctive normal forms that are equivalent?

BASIC LOGICAL EQUIVALENCES

Idempotence.	$p \vee p \equiv p$
	$p \wedge p \equiv p$
Commutativity	$p \vee q \equiv q \vee p$
	$p \wedge q \equiv q \wedge p$
Associativity	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
Distributivity	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Double negation.	$\neg(\neg p) \equiv p$
Domination.	$p \vee T \equiv T$
	$p \wedge F \equiv F$
DeMorgan's Laws.	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Implications.	$p \rightarrow q \equiv q \vee \neg p$

LOGICAL EQUIVALENCES

Show the following equivalences:

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$$

TAUTOLOGIES

Show that the following statements are tautologies.

1. $(p \land q) \longrightarrow (p \lor q)$

2. $\neg p \land (p \lor q) \longrightarrow q$

TAUTOLOGIES

Determine whether or not the following statements are tautologies.

$$(\neg p \land (p \to q)) \longrightarrow \neg q$$

$$(\neg q \land (p \to q)) \longrightarrow \neg p$$

REFLEXIVE

Which of the following are reflexive relations on the set of people in the world?

a lives within a mile of b
a is taller than b
a has the same birthday as b
a has a common grandparent with b
a lives in the same country as b

SYMMETRIC

Which of the following are symmetric relations on the set of people in the world?

a lives within a mile of b
a is taller than b
a has the same birthday as b
a has a common grandparent with b
a lives in the same country as b

TRANSITIVE

Which of the following are transitive relations on the set of people in the world?

a lives within a mile of b
a is taller than b
a has the same birthday as b
a has a common grandparent with b
a lives in the same country as b

EQUIVALENCE RELATIONS

Which of the following are equivalence relations on the set of people in the world?

a lives within a mile of b
a is taller than b
a has the same birthday as b
a has a common grandparent with b
a lives in the same country as b

For all equivalence relations, find the quotient sets.

EQUIVALENCE RELATIONS

Are the following relations on $\mathbb{Z}$ reflexive? symmetric? transitive?

$\leq$

$\neq$

$\{(x,y) \mid |x-y| \leq 1\}$

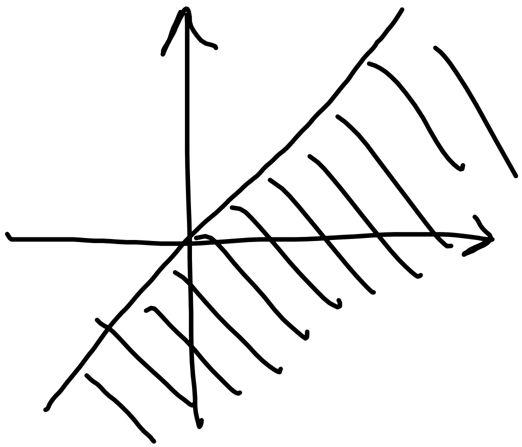
$\{(x,y) \mid x-y \text{ is divisible by } 10\}$

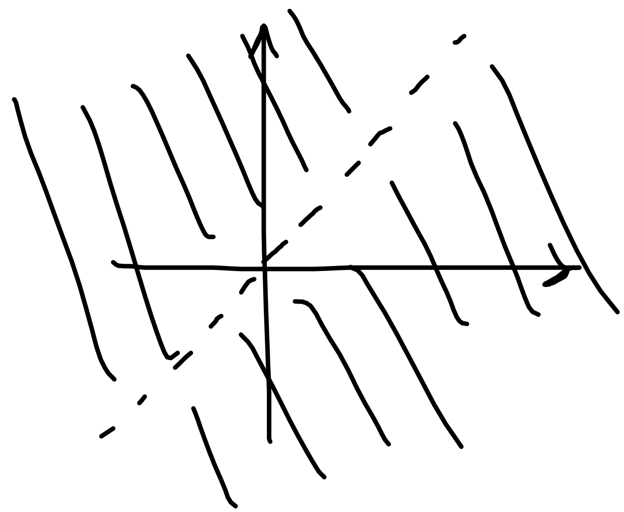
$\{(x,y) \mid x-y \text{ is divisible by } 2\}$

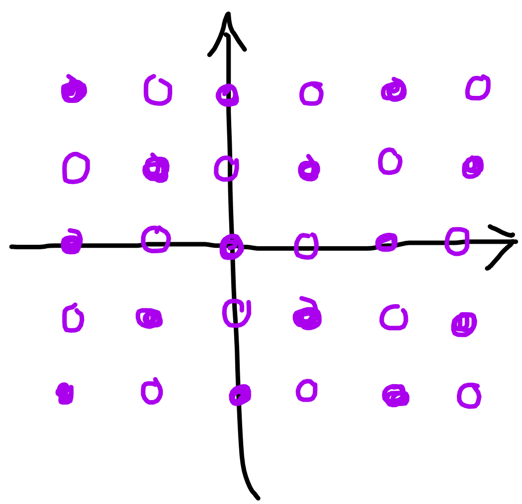
For all equivalence relations, find the quotient set.

PICTURES

What relations on $\mathbb{R}$ or $\mathbb{Z}$ are depicted?

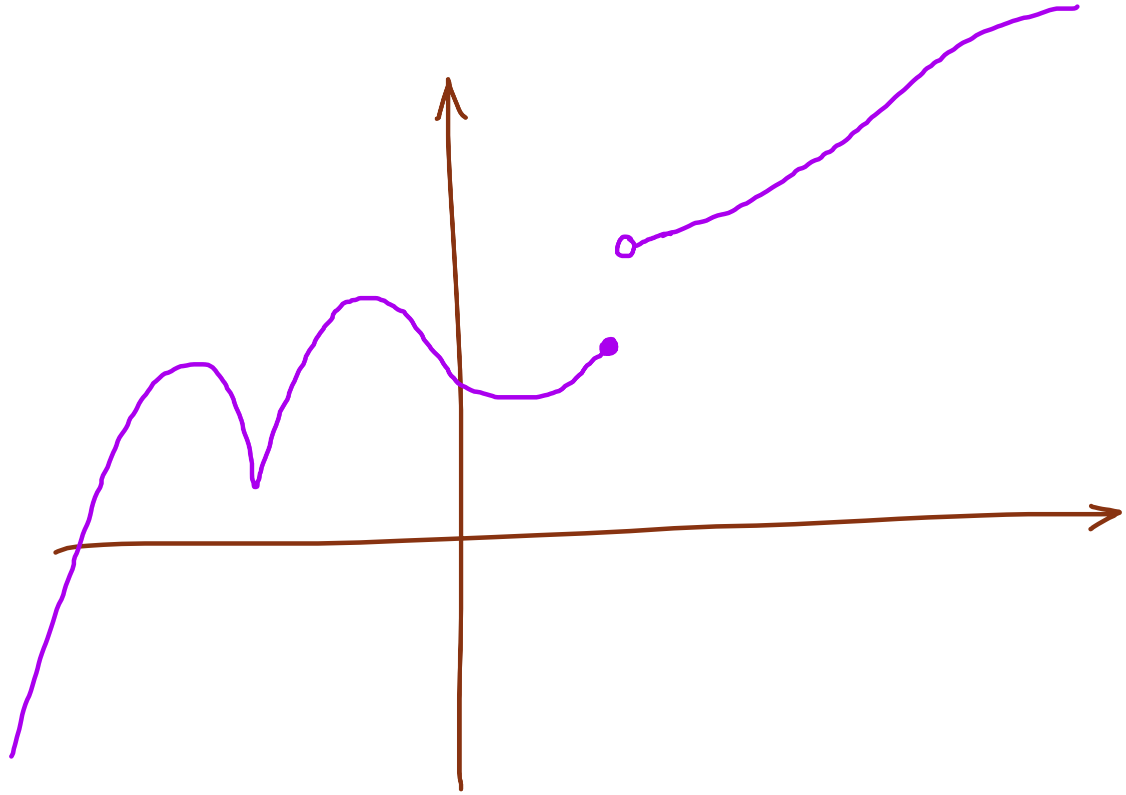






FUNCTIONS

Is this a function?



ONTO

Which of the following functions are onto?

1. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 4x + 5$
2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$
3. $f: \mathbb{R} \to \mathbb{Z}$, $f(x) = \lceil x \rceil$
4. $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x$
5. $f: \{\text{people}\} \to \{A, \dots, Z\}$, $f(x) =$ first initial of x.

When a function is not onto, describe the image.

ONE-TO-ONE

Which of the following functions are one-to-one?

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 4x + 5$

2. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$

3. $f : \mathbb{R} \to \mathbb{Z}$, $f(x) = \lceil x \rceil$

4. $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x$

5. $f : \{\text{people}\} \to \{A, \ldots, Z\}$, $f(x) =$ first initial of x.

When a function is not one-to-one, find the largest domain on which it is.

IDENTITY FUNCTION

Let A be a set. The identity function is always...

A. one-to-one
B. onto
C. both
D. neither

INVERSES

Find the inverses of the following functions.

$f(x) = x^2$

$f(x) = 3x + 7$

$f(x) = \frac{x}{x-1}$

$f(x) = \ln(2x - 5)$

COMPOSITION

Say $f(x) = \lceil x \rceil$ and $g(x) = \sqrt{x}$.

What is the domain of $f \circ g$?

What is $f \circ g\,(10)$?

Find the inverse of $f(x) = (2x+8)^3$ and check $f \circ f^{-1}$ is the identity.

HOTEL INFINITY

Hotel Infinity has rooms numbered 1, 2, 3, ...

Today, every room is occupied. Someone walks into the lobby and asks for a room. Can the hotel accommodate her?

HOTEL INFINITY

Hotel Infinity is so successful they open Hotel Infinity 2, just like the first.

There is a fire in Hotel Infinity 2. Can Hotel Infinity accommodate the overflow?

HOTEL INFINITY

Now there is Hotel Infinity 2, Hotel Infinity 3, etc. All the hotels except the first burn to the ground. Can Hotel Infinity accommodate the overflow?

HOTEL INFINITY

Show that the following sets are countable:

$\mathbb{N} \cup \{0\}$

$\mathbb{N} \cup \mathbb{N}$

$\mathbb{N} \cup \mathbb{N} \cup \mathbb{N} \cup \cdots$

INTERVALS

Show that the following pairs of sets have the same cardinalities.

$(0,1)$ and $(1,3)$

$(-\infty,\infty)$ and $(0,\infty)$

CANTOR DIAGONALIZATION

THEOREM. $\mathbb{R}$ is uncountable.

RATIONALS

Is $\mathbb{Q}$ countable?

What about $\mathbb{R} \setminus \mathbb{Q}$?

What about the Cantor set?

LINES AND SQUARES

Show that $|[0,1]| = |[0,1]^2|$.

COUNTING SOLDIERS

A general lines her troops in rows of 9, then 10, then 11. Each time, there are leftovers: 1, 2, and 4, respectively.

Can the general tell just from this information exactly how many soldiers she has?

CONGRUENCE

Compute:

$1234567 \pmod{10}$

$1027581 \pmod{2}$

$624897 \pmod{3}$

$169 \pmod{24}$

CONGRUENCE

Compute:

$$101 \times 122 \pmod{3}$$

$$4^{157} \pmod{3}$$

$$149728 \times 51 \pmod{3}$$

CONGRUENCE

Solve: $x + 7 \equiv 2 \pmod{19}$

Solve: $x \equiv 1 \pmod{3}$
$x \equiv 3 \pmod{5}$

MULTIPLICATIVE INVERSES

Does every number have a multiplicative inverse mod n ?

CONGRUENCE

Solve: $2x \equiv 1 \pmod{9}$

$7x \equiv 1 \pmod{100}$

$7x \equiv 2 \pmod{100}$

$7x \equiv 3 \pmod{100}$

CHINESE REMAINDER THEOREM

Solve: $x \equiv 1 \pmod{3}$
$x \equiv 3 \pmod{5}$

Solve: $x \equiv 1 \pmod{3}$
$x \equiv 3 \pmod{5}$
$x \equiv 2 \pmod{7}$

Solve: $x \equiv 1 \pmod{9}$
$x \equiv 2 \pmod{10}$
$x \equiv 4 \pmod{11}$

Hint: $6 \cdot 90 - 49 \cdot 11 = 1$

CHINESE REMAINDER THEOREM

Solve: $x \equiv 1 \pmod 9$
$x \equiv 2 \pmod{10}$
$x \equiv 4 \pmod{11}$

Hint: $6 \cdot 90 - 49 \cdot 11 = 1$

First solve just the first two, mod 90. Since

$$1 \cdot 10 - 1 \cdot 9 = 1$$

we have: $1 \cdot (1 \cdot 10) - 2(1 \cdot 9) = -8 \equiv 82 \pmod{90}$

Now solve: $x \equiv 82 \pmod{90}$
$x \equiv 4 \pmod{11}$

Since $6 \cdot 90 - 49 \cdot 11 = 1$

we have: $4(6 \cdot 90) - 82(49 \cdot 11) = -42{,}038 \equiv 532 \pmod{990}$

INDUCTION

Prove the following statements by induction.

(1) $\sum_{i=1}^{n} i = n(n+1)/2 \qquad n \geq 1$

(2) $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6 \qquad n \geq 1$

(3) $\sum_{i=1}^{n} (2i-1) = ?? \qquad n \geq 1$

TOWERS OF HANOI

Use induction to show that it is possible to solve the Towers of Hanoi puzzle with n disks.

INDUCTION

Prove the following statements by induction.

(1) $7^n - 1$ is divisible by 6 for all $n \geq 0$

(2) $n^2 + 2n$ is divisible by 3 for all $n \geq 0$

(3) $(2n)!$ is divisible by 2^n for $n \geq 0$.

INDUCTION

Prove the following statements by induction.

(1) $n! > 2^n \quad n \geq 4$

(2) $\frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{1}{2} \quad n \geq 1$

(3) $(1 + 1/2)^n \geq 1 + n/2 \quad n \geq 0.$

(4) $(1+x)^n \geq 1 + nx \quad n \geq 0$

INDUCTION

Prove the following statements by induction.

(1) The interior angle sum of a convex n-gon is $(n-2)\pi$.

(2) If n lines in $\mathbb{R}^2$ have no triple intersections then they divide the plane into $n+1$ regions.

(3) $F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$

OTHER INDUCTIONS

Which of the following are correct forms of induction?

(1) If $P(n_0)$ is true and $P(k+1)$ is true whenever $P(k)$ and $P(k-1)$ are true $(k > n_0)$ then $P(n)$ is true for $n \geq n_0$.

(2) If $P(n_0)$ is true and $P(k)$ is true whenever $P(n_0), \ldots, P(k-1)$ are true $(k > n_0)$ then $P(n)$ is true for all $n \geq n_0$

(3) If $P(5)$ is true and $P(k)$ is true whenever $P(k-1)$ is true then $P(n)$ is true for all $n \geq 5$.

MORE INDUCTION

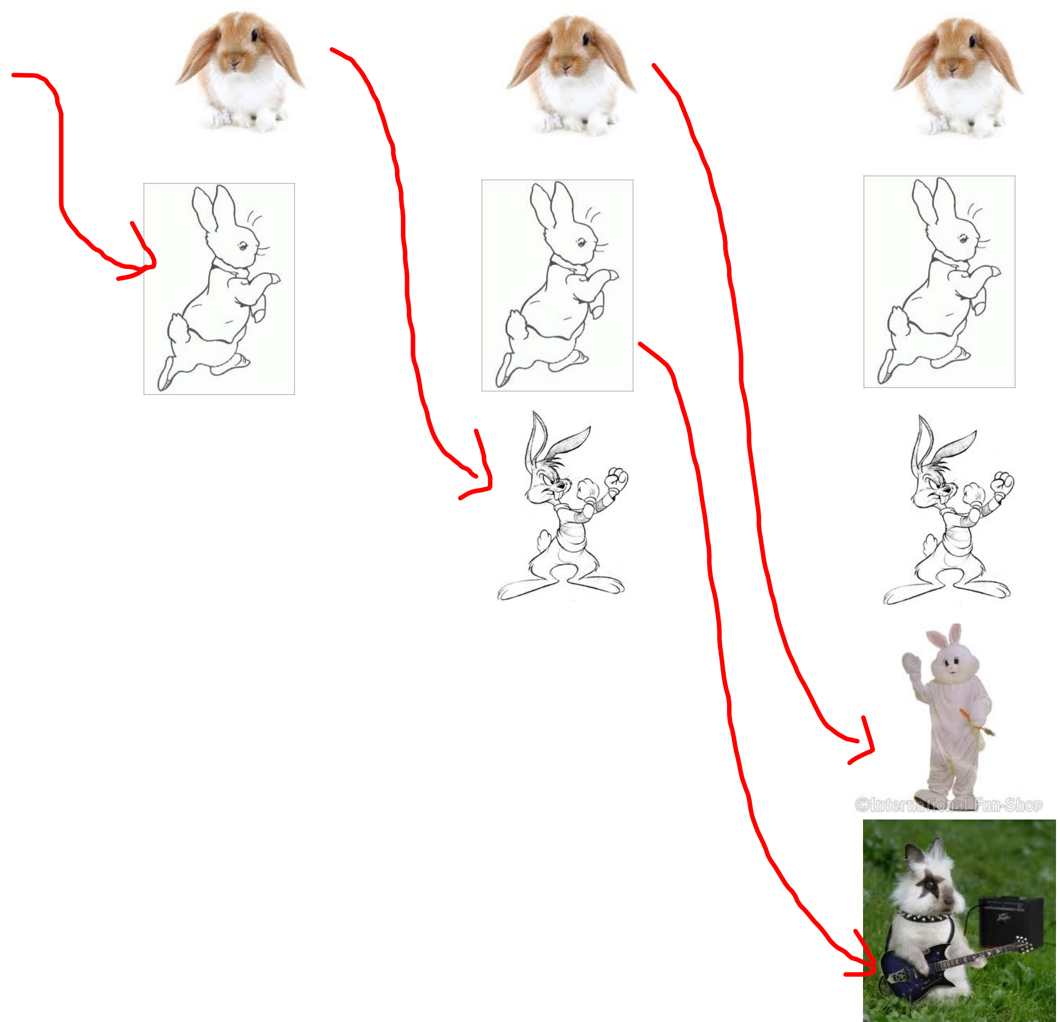
Prove the following statements by induction.

(1) Every natural number has a prime factorization.

(2) In a convex n-gon one can draw at most $n-2$ non-intersecting diagonals.

(3) The number of ways of breaking a $2 \times n$ candy bar into 2×1 pieces is F_{n+1}

BUNNIES



How many bunnies in month 10?

Month

1 2 3 4 5

Towers of Hanoi

How many moves are needed to solve the towers of Hanoi puzzle with n disks?

SOLVING RECURRENCE RELATIONS

Use induction to show that the purported solutions are really solutions.

(1) $a_n = a_{n-1} + 2, \quad a_0 = 1$

Solution: $a_n = 2n + 1$

(2) $a_n = 2a_{n-1} + 1, \quad a_0 = 1$

Solution: ??

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

Solve: $a_n = a_{n-2}$, $a_0 = 1$, $a_1 = 3$.

$a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 0$

$a_n = 2a_{n-1} + a_{n-2}$, $a_0 = 0$, $a_1 = 1$

MORE PROBLEMS

① Solve $a_n = 9a_{n-2}$ where

(a) $a_0 = 6,\ a_1 = 12$

(b) $a_0 = 6,\ a_2 = 54$

(c) $a_0 = 6,\ a_2 = 10$

② Solve $a_n = 8a_{n-1} - 16a_{n-2},\ a_0 = 1,\ a_1 = 16$

③ Solve $5a_n = 11a_{n-1} - 2a_{n-2},\ a_0 = 2,\ a_1 = -8.$

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Solve: $a_n = 2a_{n-1} + 1, \quad a_1 = 1$

$a_n = 3a_{n-1} + 5 \cdot 7^n, \quad a_0 = 2.$

$a_n = -a_{n-1} + n, \quad a_0 = 1/4.$

$a_n = 2a_{n-1} - n/3, \quad a_0 = 1$

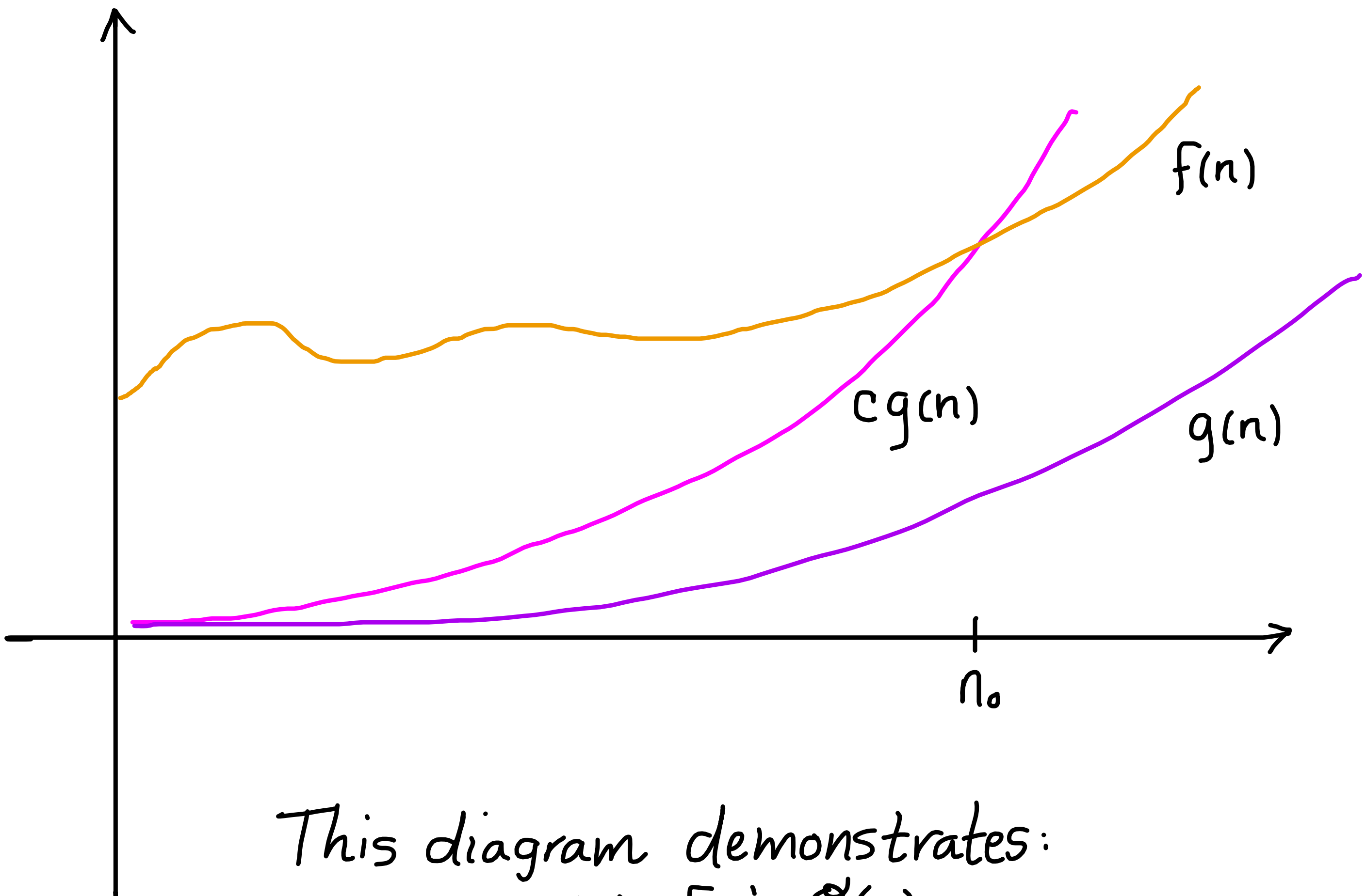
MORE PROBLEMS

① Solve $a_n = 5a_{n-1} - 6a_{n-2} + 6 \cdot 4^n$

② Solve $a_n = a_{n-1} + 3n^2$, $a_0 = 7$

By the way, there is another method for solving #2, the method of undetermined coefficients. Idea: recursively substitute: $a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum i^2 = \cdots$

BIG O



This diagram demonstrates:

(a) f is $O(g)$

(b) g is $O(f)$

(c) both

BIG O

We say that "f is big O of g" and write

$$f = O(g) \quad \text{or} \quad f \in O(g)$$

if there is a natural number n_0 and a positive real number c such that

$$|f(n)| \leq c\,|g(n)|$$

for $n \geq n_0$.

First examples: 1. $f(n) = n^2$, $g(n) = 7n^2$

2. $f(n) = 4n+2$, $g(n) = n$

3. $f(n) = n^2$, $g(n) = n^2+2n+1$

4. $f(n) = n$, $g(n) = \sqrt{n}$

LIMIT THEOREM

THEOREM: Let f,g be functions $\mathbb{N} \to [0,\infty)$

(a) If $\lim_{n\to\infty} f(n)/g(n) = 0$, then $f \prec g$

(b) If $\lim_{n\to\infty} f(n)/g(n) = \infty$, then $g \prec f$

(c) If $\lim_{n\to\infty} f(n)/g(n) = L \neq 0$, then $f \asymp g$

MORE EXAMPLES

① Compare $n!$ & n^n

② Compare $n!$ & 2^n

COMBINING FUNCTIONS

Theorem: Let f, g be functions $\mathbb{N} \to \mathbb{R}$.

(a) If $f \in \mathcal{O}(F)$, then $f + F \in \mathcal{O}(F)$

(b) If $f \in \mathcal{O}(F)$ and $g \in \mathcal{O}(G)$ then $fg \in \mathcal{O}(FG)$.

POLYNOMIALS

Theorem: Let $f(n) = a_d n^d + \cdots a_1 n + a_0$ be a degree d polynomial ($a_d \neq 0$). Then $f(n) \asymp n^d$.

MORE COMPARISONS

Theorem: (a) If $k < \ell$, then $n^k < n^\ell$

(b) If $k > 1$, then $\log_k n < n$

(c) If $k > 0$, then $n^k < 2^k$

HIERARCHY

$1 < \log n < n < n^k < k^n < n! < n^n$

const < log < linear < poly < exp < fact < tower

MORE DETAILED HIERARCHY

$$1 < \log n < \sqrt{n} < n/\log n < n < n\log n < n^{3/2}$$

$$< n^2 < n^3 < \cdots$$

$$< 2^n < 3^n < \cdots$$

$$< n!$$

$$< n^n < n^{n^n} < \cdots$$

COMPARING DIFFERENT ORDERS

	10	50	100	300	1000
$5n$	50	250	500	1500	5,000
$n \log n$	33	282	665	2469	9966
n^2	100	2500	10,000	90,000	1,000,000
n^3	1,000	125,000	1 mil	27 mil	1 bil
2^n	10^{24}	16 digits	31 dig.	91 dig.	302 dig.
$n!$	3.6 mil	65 dig.	161 dig.	623 dig.	unimaginable
n^n	10 bil.	85 dig.	201 dig.	744 dig.	unimaginable

μsecs since big bang: ~10^{24}

protons in the known universe: ~10^{126}

D. Harel, Algorithmics

COMPARING DIFFERENT ORDERS

How long would it take at 1 step per μsec?

	10	20	50	100	300
n^2	1/10,000 sec.	1/2500 sec.	1/400 sec	1/100 sec.	9/100 sec.
n^5	1/10 sec.	3.2 sec	5.2 min	2.8 hr	28.1 days
2^n	1/1,000 sec	1 sec	35.7 yr	400 trillion cent.	75 digit # of centuries
n^n	2.8 hr	3.3 trillion yr	70 digit # of centuries	185 digit # of centuries	728 digit # of centuries.

D. Harel, Algorithmics

PHONE NUMBERS

Are there two students at Georgia Tech with the same last 4 digits of their phone number?

HAIR

Are there two non-bald people in Atlanta with the same number of hairs on their heads?

THE PIGEONHOLE PRINCIPLE

If n objects are put into m boxes, and $n > m$, then at least one box will have multiple objects.

Johann Peter Gustav Lejeune Dirichlet

PIGEONHOLE PROBLEMS

1. Show that, given 5 points in a unit square, there are two points within $\sqrt{2}/2$ of each other.

2. Show that, given any 11 integers, there is a pair of numbers whose difference is divisible by 10.

3. Show that, at any party, there are always two people with the same number of friends.

PIGEONHOLE PROBLEMS

4. Take a chessboard with two opposite corners removed. Can you cover it with dominos?

   Hint: The dominos give a bijection between black squares and white squares.

5. On a 5×5 chessboard, there is one flea in each square. Each flea jumps to an adjacent square. Are there now two fleas in the same square?

6. Arrange the numbers $1, \ldots, 10$ on a circle in any order. Show that there are 3 consecutive numbers that add to 17 or more.

STRONG PIGEONHOLE

Our class has 68 students. What is the biggest N so that we know that some month has N birthdays?

The challenge with counting is that we aren't usually told in advance which rules to use

MORE PROBLEMS

1. How many 3 digit numbers are there?

2. How many 3 digit numbers are there with no repeated digits?

3. How many 3 digit numbers are there with the i^{th} digit equal to i for some i.

MORE PROBLEMS

4. How many functions are there $A \to B$ if $|A| = m$, $|B| = n$?

5. How many injective functions are there $A \to B$ if $|A| = m$, $|B| = n$?

6. How many subsets of A are there if $|A| = n$?

MORE PROBLEMS

7. How many even 4 digit numbers are there with no repeated digits?

8. How many odd 4 digit numbers are there with no repeated digits? (Harder!)

9. How many ways are there to place a domino on a chessboard?

MORE PROBLEMS

10. How many bit strings are there that have length n and begin and/or end with a 1?

11. How many different dominos are there?

12. How many arrangements are there of 6 men and 4 women at a round table if no women sit together?

MORE PROBLEMS

13. Given 20 integers, show there is a pair whose difference is divisible by 19.

14. If we want to label the chairs in a room by one letter and one number from 1 to 100, how many labels are there?

15. How many distinct alphanumeric passcodes are there if each passcode has 6-8 characters and at least one digit?

16. In how many ways can a best-of-5 series go down?

17. Given 5 points on a sphere, how many necessarily lie on the same hemisphere?

Permutations

In a club with 10 people, how many ways are there to choose a president, vice president, and secretary?

PERMUTATIONS

How many permutations of 4 objects?

PERMUTATION PROBLEMS

A group has n men and n women. In how many ways can they be lined up so that men and women alternate?

PERMUTATION PROBLEMS

How many ways are there to seat 6 boys and 4 girls at a round table if no two girls sit together?

Note: A rotation of a configuration is considered the same as the original configuration.

PERMUTATION PROBLEMS

Arrange all 26 letters of the alphabet in a row.

a) How many such "words" are there?

b) How many contain HAMLET as a subword, e.g.:

VRPKGCHAMLETBDFIZWJNQOSYUX

c) How many have exactly 4 letters between H and T?

COMBINATIONS

In a club with 10 people, how many ways to choose a committee with 3 members?

MARBLES AND BOXES

Distinguishable marbles: Say we want to put a red, a green, and a blue marble into 5 boxes.
How many ways?

Indistinguishable marbles: Say we want to put 3 indistinguishable marbles in 5 boxes.
How many ways?

COMBINATION PROBLEMS

1. Five people need a ride. My car holds 4. In how many ways can I choose who gets a ride?

2. If you toss a coin 7 times, in how many ways can you get 4 heads?

3. The House of Representatives has 435 representatives. How many 4-person committees can there be?

MORE PROBLEMS

1. How many bit strings are there with fifteen 0's and six 1's if every 1 is followed by a 0?

Note: Too hard if you think of it as a sequence of 21 tasks.

MORE PROBLEMS

2. How many strings in the letters a, b, and c have length 10 and exactly 4 a's?

Again, don't choose the 10 letters one by one.

MORE PROBLEMS

3. A lottery ticket has six numbers from 1 to 40. How many different tickets are there?

The lottery agency chooses six winning numbers. How many different possible lottery tickets have exactly four winning numbers?

MORE PROBLEMS

4. Determine the number of alphabetic strings of length 5 consisting of distinct (capital) letters that

   (a) do not contain A
   (b) contain A
   (c) start with ABC
   (d) start with A, B, C in any order
   (e) contain A, B, C in that order
   (f) contain A, B, C

MORE PROBLEMS

5. Determine the number of possible softball teams (= 9 people) can be made from a group of 10 men, 12 women, and 17 children if:
   (a) there are no restrictions
   (b) there must be 3 men, 3 women, 3 children
   (c) the team must be all men, all women, or all children
   (d) the team cannot have both men and women.

MORE PROBLEMS

6. In how many ways can you put 5 indistiguishable red balls and 8 indistinguishable green balls into 20 boxes if
   (a) there can be at most one ball per box
   (b) there can be at most one ball of each color per box.

MORE PROBLEMS

7. How many poker hands are:

(a) total
(b) 4 of a kind
(c) flush
(d) straight
(e) straight flush
(f) full house
(g) 3 of a kind
(h) 2-pair
(i) pair
(j) neither flushes straights, full house 3 of a kind, 2 pair, pair

PROBABILITY

You have 3 cards. One is red on both sides, one is black on both sides, and one has a red side and a black side. You pick one card randomly and put it on the table. Its top side is red. What is the probability the other side is red?

PROBABILITY

What is the probability that...

a) A flipped coin comes up heads?

b) A rolled die comes up 3?

c) A rolled pair of dice comes up 4?

MORE EXAMPLES

1. You toss a coin 5 times. What is the probability of getting 4 heads?

2. What is the probability of correctly guessing the winners in a 64-team single elimination tournament?
(Assume every team has a 50% chance of winning each game)

MORE EXAMPLES

3. An urn has 4 red balls, 3 green balls. You pull one ball at random. What is the probability of pulling a green ball?

Suppose you pull one ball, replace it, then pull another ball. What is the probability of pulling two balls of the same color?

MORE EXAMPLES

Same urn (4 red, 3 green). Now suppose you pull one ball, don't replace it, and pull another ball. What is the probability of getting two balls of the same color?

MORE EXAMPLES

4. In poker, what is the probability of dealing a 4-of-a-kind?

What about a full house?

APPLYING PROBABILITY RULES

EXAMPLE: A number from 1 to 100 is chosen at random. What is the probability it is...

a) divisible by 2, 3, or 5?

b) divisible by 2 and 3, but not 5?

c) divisible by 3 but not 2 or 5?

d) divisible by at most two of 2, 3, and 5?

MUTUAL EXCLUSIVITY

Two events A and B are mutually exclusive if $A \cap B = \emptyset$

Events $A_1, \ldots, A_n$ are pairwise mutually exclusive if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

If $A_1, \ldots, A_n$ are pairwise mutually exclusive events, then
$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n) \quad \text{(addition rule)}$$

EXAMPLE: A number from 1 to 100 is chosen at random. What is the probability that the number is divisible by 7 or 30?

APPLYING PROBABILITY RULES

1. What is the probability that a length 10 bit string (chosen at random) has at least one zero? at least two zeros?

2. What is the probability that a poker hand (dealt at random) is a flush? a straight? royal flush?

Note: A, 2, 3, 4, 5 and 10, J, Q, K, A are both straights.

THE MONTY HALL PROBLEM

"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to take the switch?"

CONDITIONAL PROBABILITY

A coin is flipped twice. The first flip is heads. What is the probability that both flips are heads?

Intuition:

Basic probability:

Conditional probability:

CONDITIONAL PROBABILITY

I have two kids. One is a boy. What is the probability I have two boys?

CONDITIONAL PROBABILITY

1. An urn has 10 white, 5 yellow, and 10 black marbles. A marble is chosen at random. We are told it is not black. What is the probability it is yellow?

2. We deal bridge hands at random to N, S, E, W. Together, N and S have 8 spades. What is the probability that E has 3 spades?

CONDITIONAL PROBABILITY

Alice and Bob each roll a die. We are told that Alice rolled a higher number. What is the probability that Alice rolled a 3?

INDEPENDENCE

Events A and B are independent if

$$P(B|A) = P(B)$$

Since $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$ we can say A and B are independent if:

$$P(A \cap B) = P(A)P(B)$$

Examples. 1. We roll two die. A = first comes up 2
B = second comes up 3

2. Two kids. B = 2 boys
A = at least one boy

INDEPENDENCE

Events A and B are independent if

$$P(B|A) = P(B)$$

Examples. 3. The Alice and Bob problem:

B = Alice rolled 3

A = Alice > Bob

4. Urn problem: 10 white, 5 yellow, 10 black.
   Are Y and B^c independent?

A CONDITIONAL PROBABILITY PROBLEM

We buy light bulbs from suppliers A and B.

30% of the bulbs come from A, 70% from B.

2% of the bulbs from A are defective

3% of the bulbs from B are defective.

What is the probability that a random bulb...

(i) is from A and defective?

(ii) is from B and not defective?

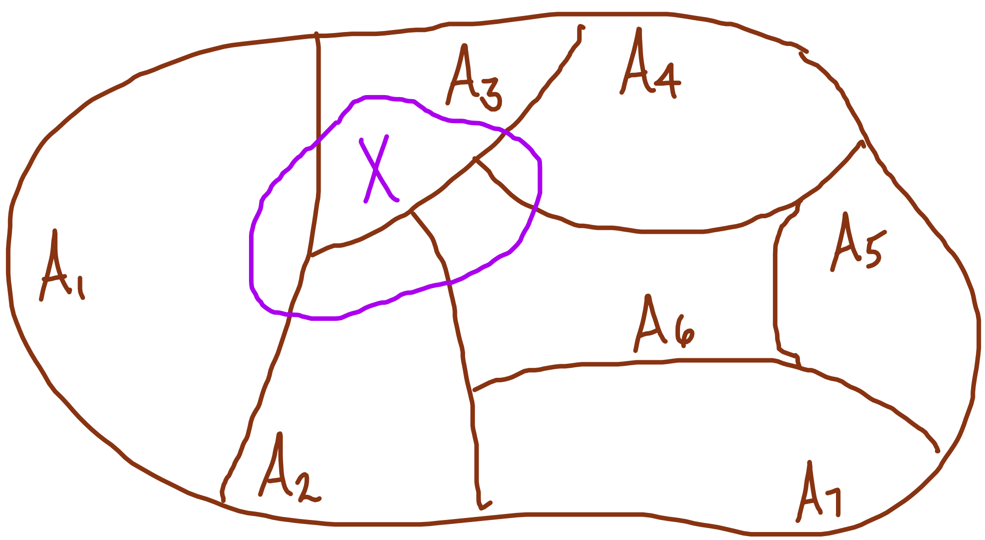
(iii) is defective?

LAW OF TOTAL PROBABILITY

Say that events $A_1, \dots, A_n$ form a partition of the sample space S, that is, the A_i are mutually exclusive ($A_i \cap A_j = \emptyset$ for $i \neq j$) and $A_1 \cup \cdots \cup A_n = S$.

Let $X \subseteq S$ be any event. Then

$$P(X) = P(A_1)P(X|A_1) + \cdots + P(A_n)P(X|A_n)$$



BAYES' FORMULA

How is $P(A|B)$ related to $P(B|A)$?

THEOREM: $$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$

PROOF: $\blacksquare$

EXAMPLE. In the light bulb problem, say a randomly selected light bulb is defective. What is the probability it came from A?

BAYES' FORMULA

EXAMPLE. Coin A comes up heads 1/4 of the time.
Coin B comes up heads 3/4 of the time.
We choose a coin at random and flip it twice.
If we get two heads, what is the probability coin B was chosen?

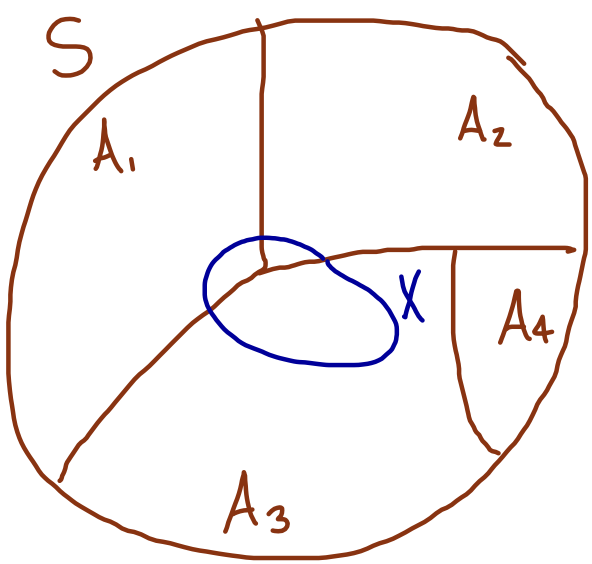
BAYES' FORMULA

Computing the denominator with the law of total probability

$A_1, \dots, A_n$ pairwise mutually exclusive events with $A_1 \cup \dots \cup A_n = S$ and $P(A_i) > 0$ for all i. Let X be an event with $P(X) > 0$. Then, for each j, we have:

$$P(A_j \mid X) = \frac{P(A_j)P(X \mid A_j)}{P(X)}$$

where $P(X) = P(A_1)P(X \mid A_1) + \cdots + P(A_n)P(X \mid A_n)$



$P(A_3 \mid X)$ big
$P(A_2 \mid X)$ small
$P(A_4 \mid X) = 0$.

EXAMPLE. Do a variant of the coin problem with 3 or more coins.

BAYES' FORMULA

PROBLEM. You have 3 cards. One is red on both sides, one is black on both sides, and one has a red side and a black side. You pick one card randomly and put it on the table. Its top side is red. What is the probability the other side is red?

BAYES' FORMULA

PROBLEM. There are 3 urns, A, B, and C that have 2, 4, and 8 red marbles and 8, 6, and 2 black marbles, respectively. A random card is picked from a deck. If the card is black we choose a marble from A, if it is a diamond we choose a marble from B, and otherwise choose a marble from C.

(a) What is the probability that a red marble gets drawn?

(b) If we know a red marble was drawn, what is the probability the card was hearts? diamonds?

Draw the picture!

REPETITIONS

QUESTION: How many ways are there to put r identical marbles into n boxes, if you are allowed to put more than one marble per box?

First try 3 marbles into 10 boxes.

Case 1: All in same box $\binom{10}{1}$

Case 2: Two in one box, one in another $10 \cdot 9$

Case 3: All different boxes $\binom{10}{3} = 120$

Addition rule $\rightsquigarrow 120 + 90 + 10 = 220.$

What about 10 marbles in 3 boxes?

Lots of cases!

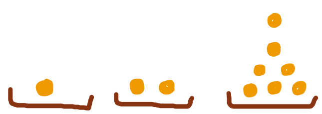
What to do?

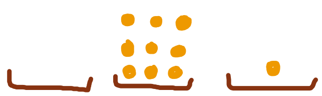
Stars and Bars

Can answer the last question by looking at it the right way:

The number of ways of putting 10 marbles into 3 boxes is the same as:

the number of binary strings with 10 zeros, 2 ones
(or 10 stars, 2 bars)

 $\longleftrightarrow$ $\star \mid \star\star \mid \star\star\star\star\star\star\star$

 $\longleftrightarrow$ $\mid \star\star\star\star\star\star\star\star\star \mid \star$

How many such strings are there?

$\binom{12}{2} = 66$ (choose which of the 12 spots will be stars.)

Repetitions

Question: How many ways are there to put r identical marbles into n boxes, if you are allowed to put more than one marble per box?

Answer: This is the same as the number of strings with r stars and $n-1$ bars:

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Repetitions, Permutations, and Combinations

How many ways to put r marbles in n boxes if...

	the marbles are indistinguishable	the marbles are distinguishable
at most one marble is allowed per box	$\binom{n}{r}$	$P(n,r)$
any number of marbles is allowed in a box	$\binom{n+r-1}{r}$	n^r

Repetitions

EXAMPLE: How many ways are there to choose 15 cans of soda from a cooler with (lots of) Coke, Dr. Pepper, Mtn Dew, RC cola, and Mr. Pibb?

FURTHER: What if I insist on at least 3 Cokes and exactly one Mr. Pibb?

REPETITIONS

EXAMPLE. In how many ways can we choose 4 nonnegative integers a, b, c, and d so that $a+b+c+d=100$?

What if a, b, c, and d are natural numbers?

REPETITIONS

EXAMPLE. How many ways are there to choose 4 integers a, b, c, and d so that:

$$a+b+c+d=15$$

$$a \geq -3,\ b \geq 0,\ c \geq -2,\ d \geq -1\ ?$$

Generalized Permutations

Example. How many ways are there to arrange the letters of SYZYGY?

Example. What about MISSISSIPPI?

Generalized Permutations

In general, say we have n objects that fall into k groups, with n_i objects in the i^{th} group. Two objects in the same group are indistinguishable, but objects in different groups are distinguishable. In how many ways can we order the objects?

$$P(n; n_1, \dots, n_k) = n!/n_1!\,n_2! \cdots n_k!$$

This is also the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in $(x_1 + x_2 + \cdots + x_k)^n$

Generalized Permutations

Example. Suppose there are 100 spots in the showroom of a car dealership. There are 15 (identical) sports cars, 25 compact cars, 30 station wagons, and 20 vans. In how many ways can the cars be parked?

MORE PROBLEMS

1. How many numbers less than 1,000,000 have the sum of their digits equal to 19?

2. A shelf holds 12 books. How many ways to choose 5 books so no adjacent books are chosen?

3. You want to visit 5 towns twice each, but there is one town you don't want to visit twice in a row. How many different travel itineraries are there?

The Binomial Theorem

THEOREM. For any x and y and any natural number n, we have:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n} x^0 y^n$$

The Binomial Theorem

Problem. Expand $(2x^3+y)^5$ and simplify.

Problem. Expand $\left(x-\frac{1}{x}\right)^6$ and simplify.

Problem. Find the coefficient of x^{15} in $\left(x^2-\frac{x}{3}\right)^{11}$.

Pascal's Triangle

THEOREM. The k^{th} entry in the n^{th} row of Pascal's triangle is $\binom{n}{k}$ for $n \geq 0$ and $0 \leq k \leq n$.

Note: The top row is considered to be row 0, and the leftmost entry is entry 0.

PROOF.

Pascal's Triangle

What is the sum of the entries in the n^{th} row?

$$1 =$$
$$1+1 =$$
$$1+2+1 =$$
$$1+3+3+1 =$$
$$1+4+6+4+1 =$$
$$\vdots$$
$$.$$

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

plug in...	to prove...
$x=1,\ y=-1$	Inclusion-exclusion principle
$x=10,\ y=1$	n^{th} row of P's Δ $= 11^k$
$x=1,\ y=1$	n^{th} row sum of P's Δ $= 2^n$
$x=\sqrt{2},\ y=-1$	$\sqrt{2}$ is irrational

The Inclusion-Exclusion Principle

Theorem. $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$

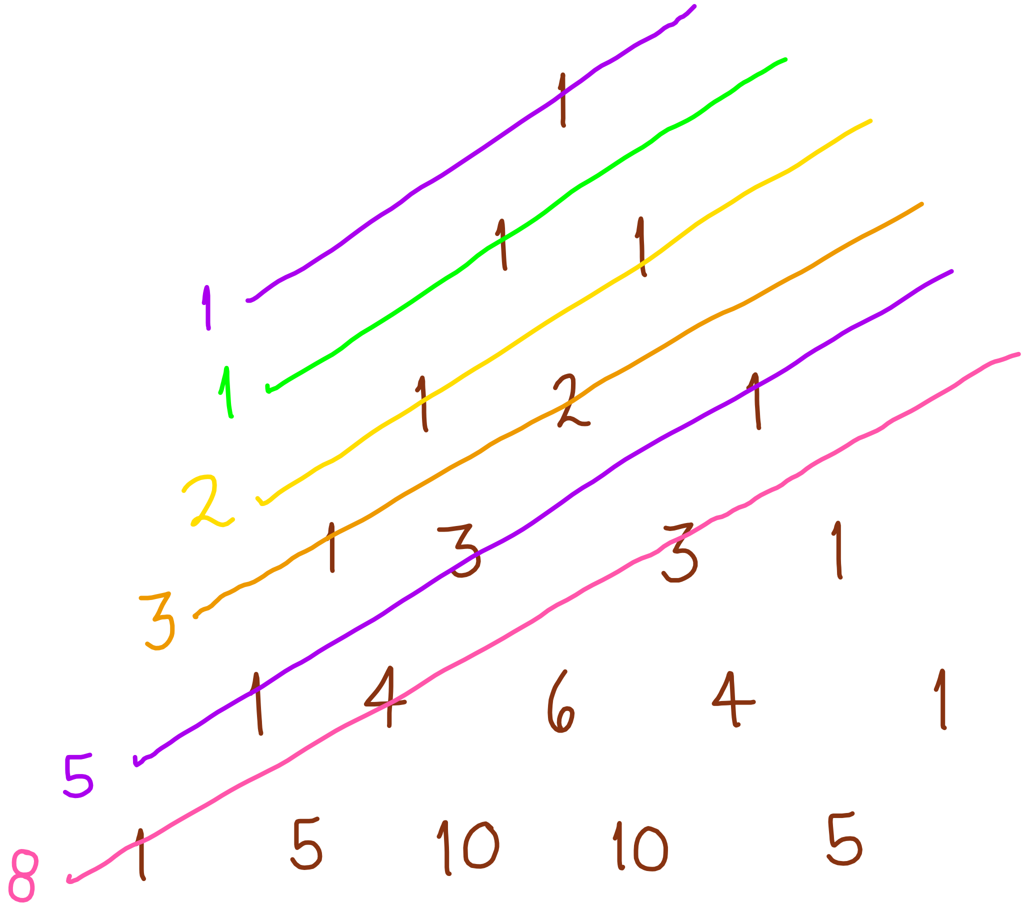
Proof:

Row Sums in Pascal's Triangle

Theorem. The sum of the entries in the n^{th} row of Pascal's triangle is 2^n.

Proof.

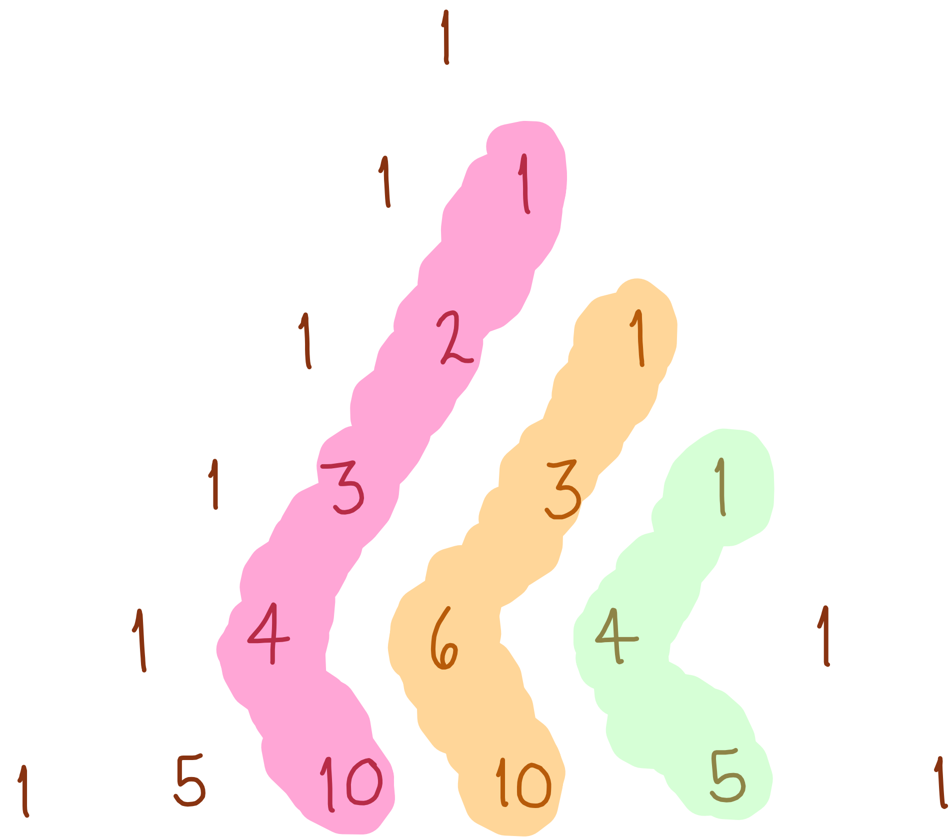
The Fibonacci Numbers In Pascal's Triangle



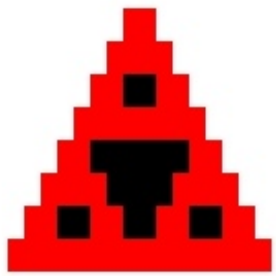
Theorem.

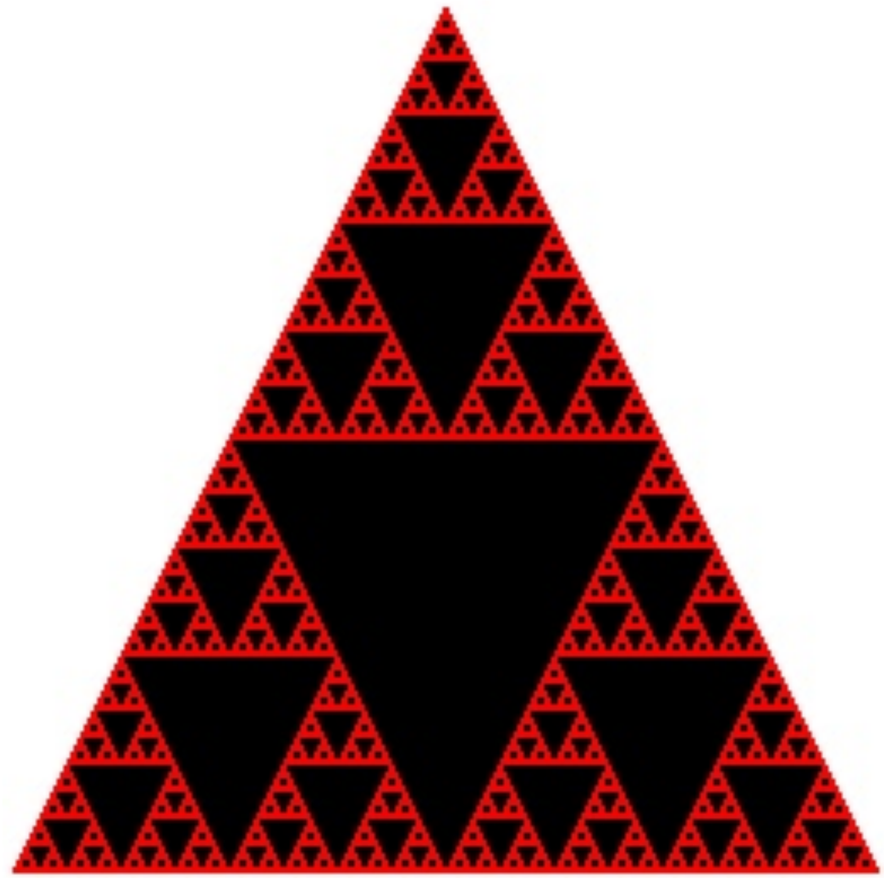
$$F_n = \begin{cases} \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{k}{k-1} & \text{if } n = 2k \\ \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{k}{k} & \text{if } n = 2k+1 \end{cases}$$

The Hockey Stick Theorem



Pascal's Triangle Mod 2





What about mod 3?

A Curious Probability

Question. A professor hands back exams randomly. What is the probability that no student gets their own exam?

Answer.

5 students ~

10 students ~

100 students ~

DERANGEMENTS

A derangement of n objects that have some natural order is a rearrangement of the objects so that no object is in its correct position.

QUESTION. How many are there? Call the number D_n.

n	D_n	$P(D_n)$
1		
2		
3		
4		

What is the pattern?

A Formula for D_n

Let A_k be the permutations of n ordered objects with object k in the correct spot.

$$D_n = \left(\bigcup_{k=1}^{n} A_k\right)^c$$

$D_4 =$

$D_4 =$

Theorem. $D_n =$

D_N and e

Theorem. $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

Recall: $e^x =$

$\rightsquigarrow e =$

$e^{-1} =$

$\approx$

So $D_n \approx$

$\rightsquigarrow P(D_n) \approx$

DERANGEMENTS

PROBLEM. Fifteen people check coats at a party and at the end they are handed back randomly. How likely is it that...

(a) Tim gets his coat back?
(b) Jeremy gets his coat back?
(c) Jeremy and Tim get their coats back?
(d) Jeremy and Tim get their coats back but no one else does?
(e) The members of the Beatles get the right set of coats back (maybe not in the right order)?
(f) Everyone gets their coat back?
(g) Exactly one person gets their coat back?
(h) Nobody gets their own coat back?
(i) At least one person gets their coat back?

A Sample Problem

Among you, your buddy, two mothers, and two sisters, some people hug. There are no hugs between buddies, mothers, or sisters. The other 5 people tell you they all hugged different numbers of people. How many people did you hug?

Four Problems

The Bridges of Konigsberg

Three House-Three Utility

Four Color

Traveling Salesman

Graphs

A graph is a pair of sets V and E, where $V \neq \emptyset$ and each element of E is a pair of elements of V.

Write $G = G(V, E)$.

The elements of V and E are called vertices and edges.

Example. V = Facebook users
E = Friendships

The Handshaking Lemma

Proposition. The sum of the degrees of the vertices of a pseudograph is an even number. Specifically:

$$\sum_{v \in V} \deg v = 2|E|$$


Leonhard Euler

Handshaking Lemma. The number of odd degree vertices of a pseudograph is even.

Proof.

Revisit the hugging problem.

The Handshaking Lemma

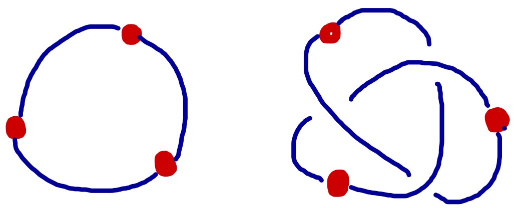
Problem. A graph has 50 edges, 4 vertices of degree 2, 6 of degree 5, 8 of degree 4, all other vertices have degree 6. How many vertices does the graph have?

Problem. Out of 24 curling players, 78 pairs have played on the same team. Show that one has played on the same team as 7 others. Show that one has played on the same team with no more than 6 others.

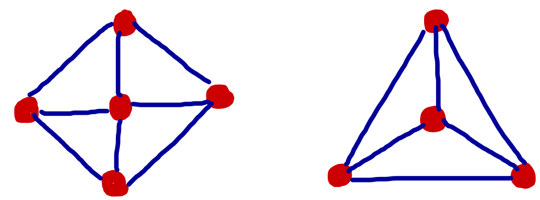
Graph Isomorphism

Which of the following pairs are isomorphic?

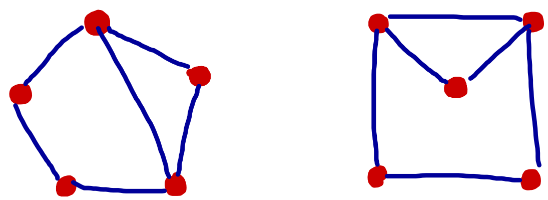
a)



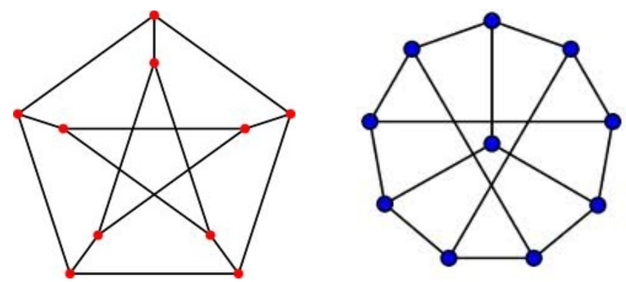
b)



c)



d)



Invariants of Graphs

We can use the following "fingerprints" of graphs in order to tell if two graphs are different:

(i) Number of vertices
(ii) Number of edges
(iii) Degree sequence
etc.

It is possible for two graphs to have the same degree sequence and be nonisomorphic:

INVARIANTS OF GRAPHS

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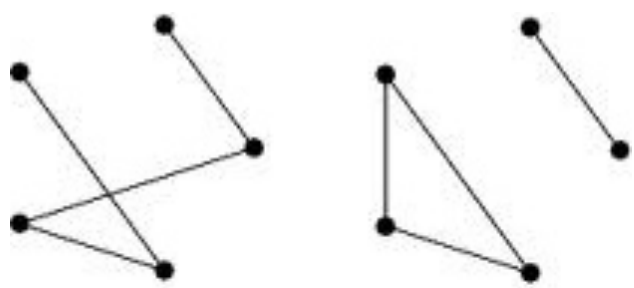
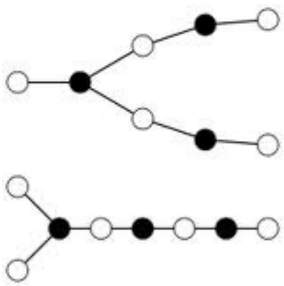


$\{2, 2, 2, 1, 1\}$ $\{2, 2, 2, 1, 1\}$

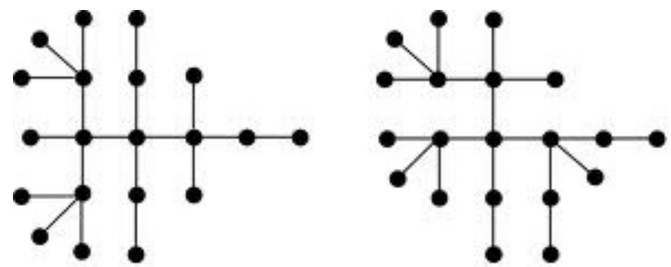
EXAMPLES

Which of the following graphs are isomorphic?

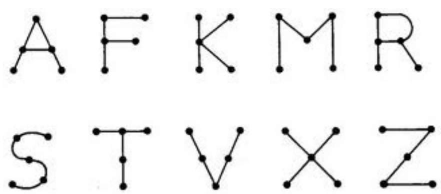
a)



b)



c)



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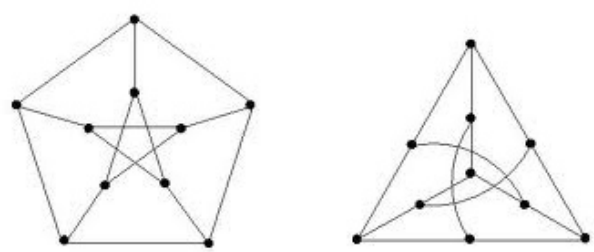
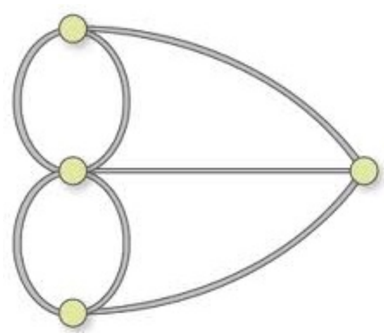


The Königsberg Bridge Problem

The Bridges of Konigsberg

Is it possible to take a walk, cross each bridge exactly once, and return to where you started?

Or: Is the following pseudograph Eulerian?

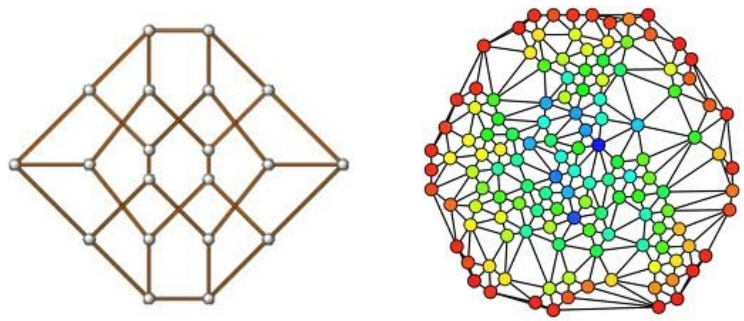


Connectivity

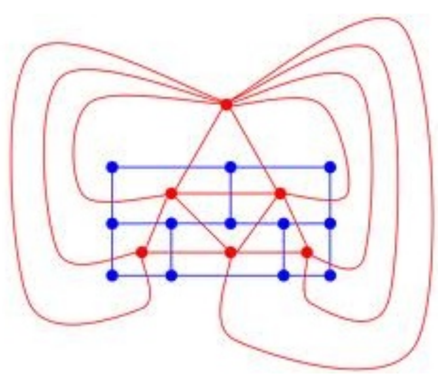
We just argued that Eulerian graphs have no vertices of odd degree.

What else? Eulerian graphs must also be connected.

A pseudograph is connected if there is a walk between any two vertices.



connected



not connected

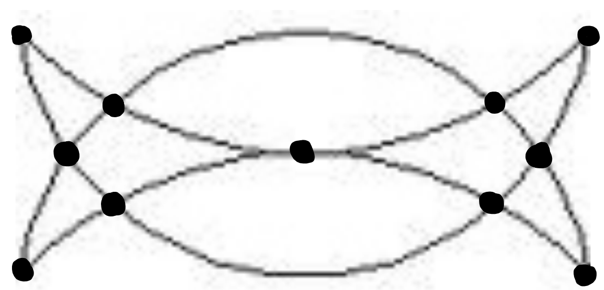
Eulerian Pseudographs

THEOREM. A pseudograph is Eulerian if and only if it is connected and every vertex has even degree.

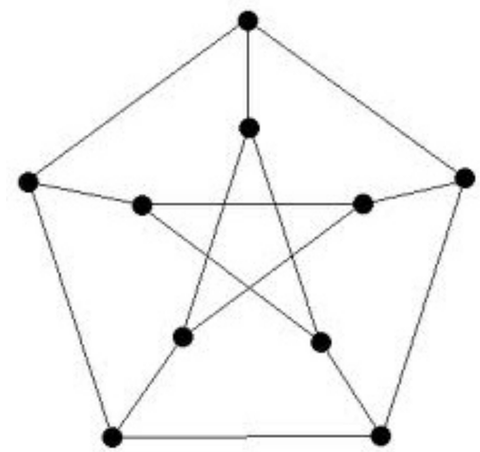
EULERIAN PSEUDOGRAPHS

For each pseudograph, find an Eulerian circuit if it exists.

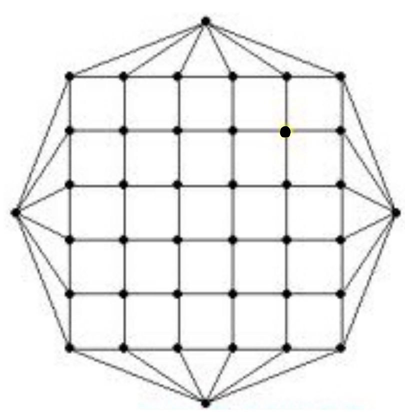
(i)



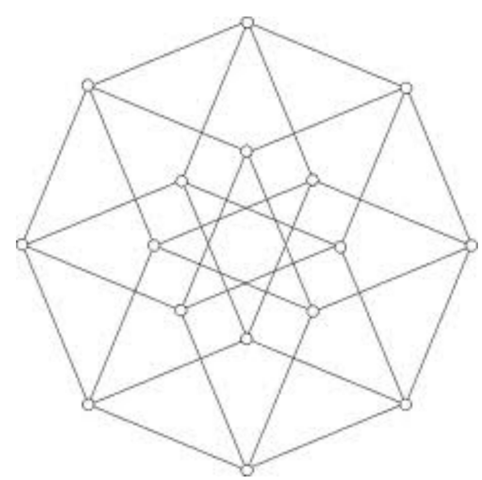
(ii)



(iii)

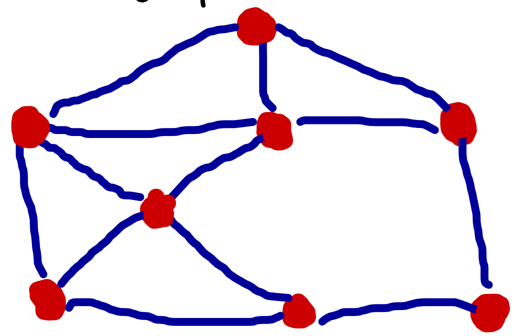


(iv)



HAMILTONIAN CYCLES

A Hamiltonian cycle in a pseudograph is a walk that visits each vertex exactly once:



Find one!

If a pseudograph has a Hamiltonian cycle, we say the pseudograph is Hamiltonian.

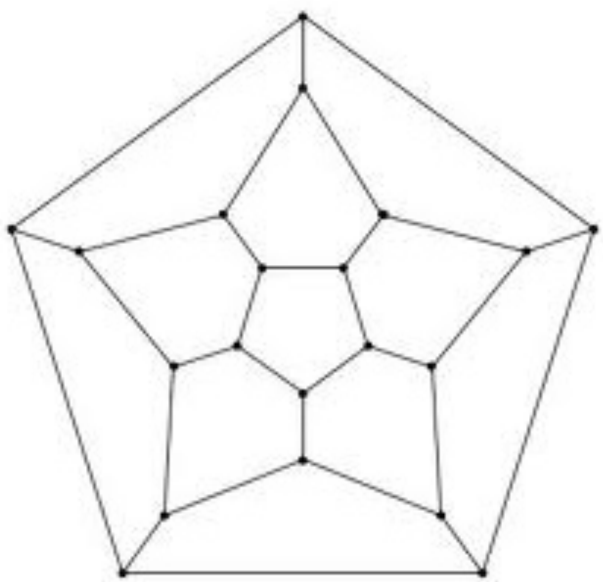
Euler: each edge once
Hamilton: each vertex once

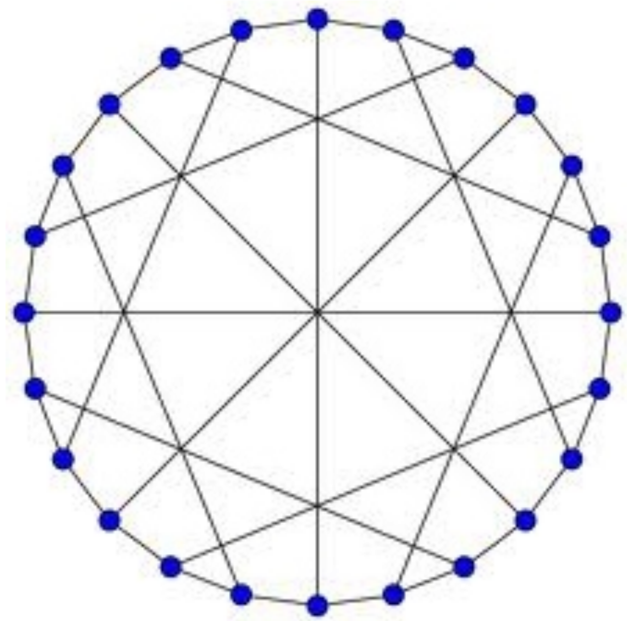
Sir William Rowan Hamilton

Note: A Hamiltonian cycle is isomorphic to an n-cycle.

Hamiltonian Cycles

Show that the following graphs are Hamiltonian.





In other words, find a Hamiltonian cycle in each.

HAMILTONIAN GRAPHS

We saw that it is easy to tell if a graph is Eulerian or not.
To prove a graph is Hamiltonian, just find a Hamiltonian cycle.
But there is no easy method for showing a graph is not Hamiltonian.

You could check all paths of length $|V|$. Takes too long!

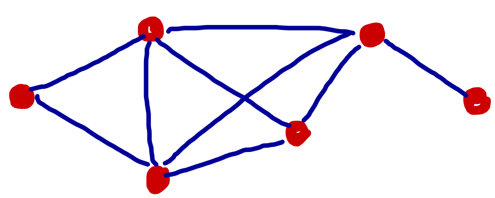
Better to use some basic facts:

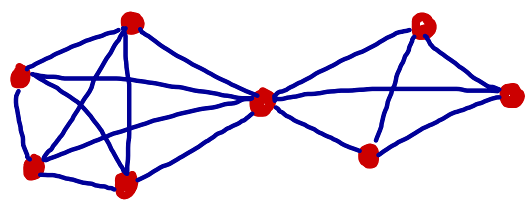
Let $\mathcal{H}$ be a Hamiltonian cycle in a pseudograph G

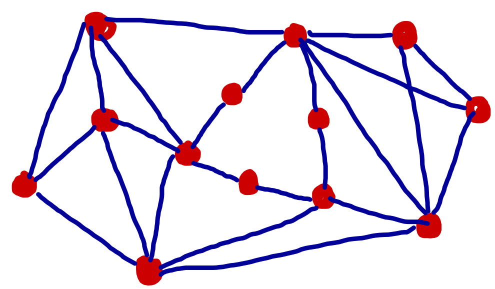
1. Every vertex of G has exactly two edges of $\mathcal{H}$ passing through it.
2. The only cycle contained in $\mathcal{H}$ is $\mathcal{H}$.

Hamiltonian Graphs

Prove that the following graphs are not Hamiltonian.

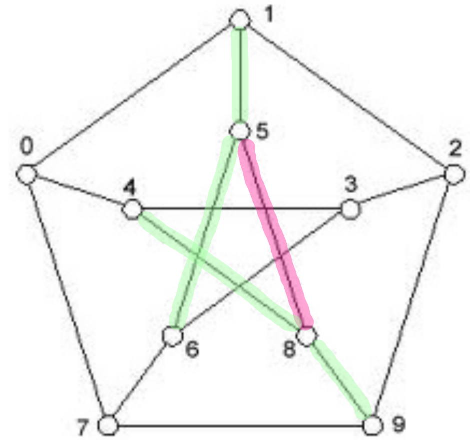






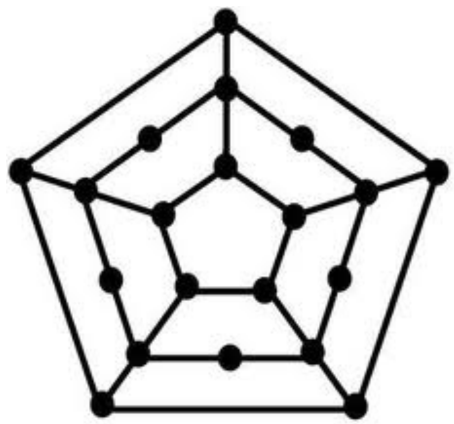
THE PETERSEN GRAPH

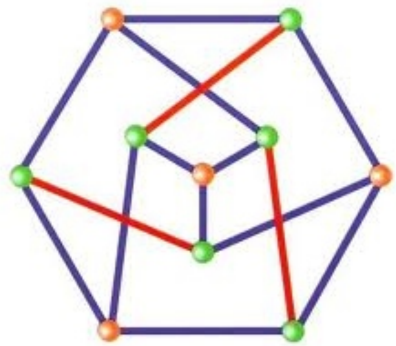
PROPOSITION. The Petersen graph is not Hamiltonian.

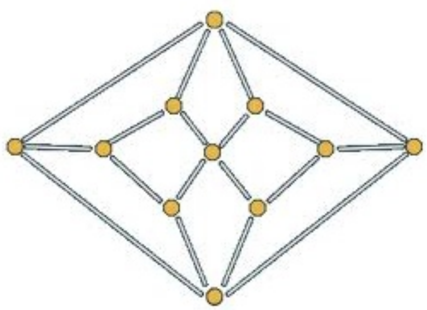


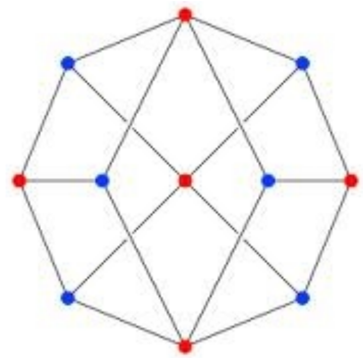
Hamiltonian Graphs

Which of the following graphs are Hamiltonian?









Hamiltonian Graphs

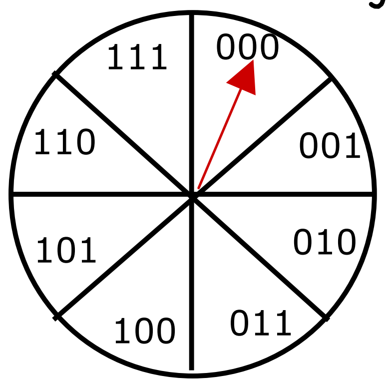
Which K_n are Hamiltonian?

Which $K_{m,n}$ are Hamiltonian?

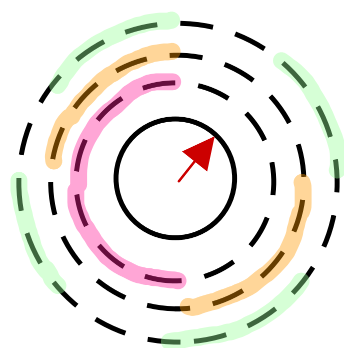
What about the knight graph on a chessboard?

Gray Codes

We can record the position of a rotating pointer with a bit string:



Can read the position of the arrow with 3 sets of contacts:

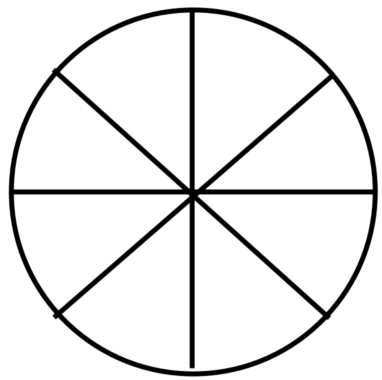


Problem: A small error could give 100 instead of 011

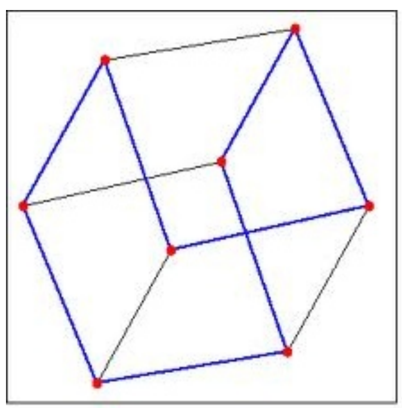
$\rightsquigarrow$ all 3 bits wrong!

Gray Codes

To fix this, want to number so that adjacent regions differ by one bit.



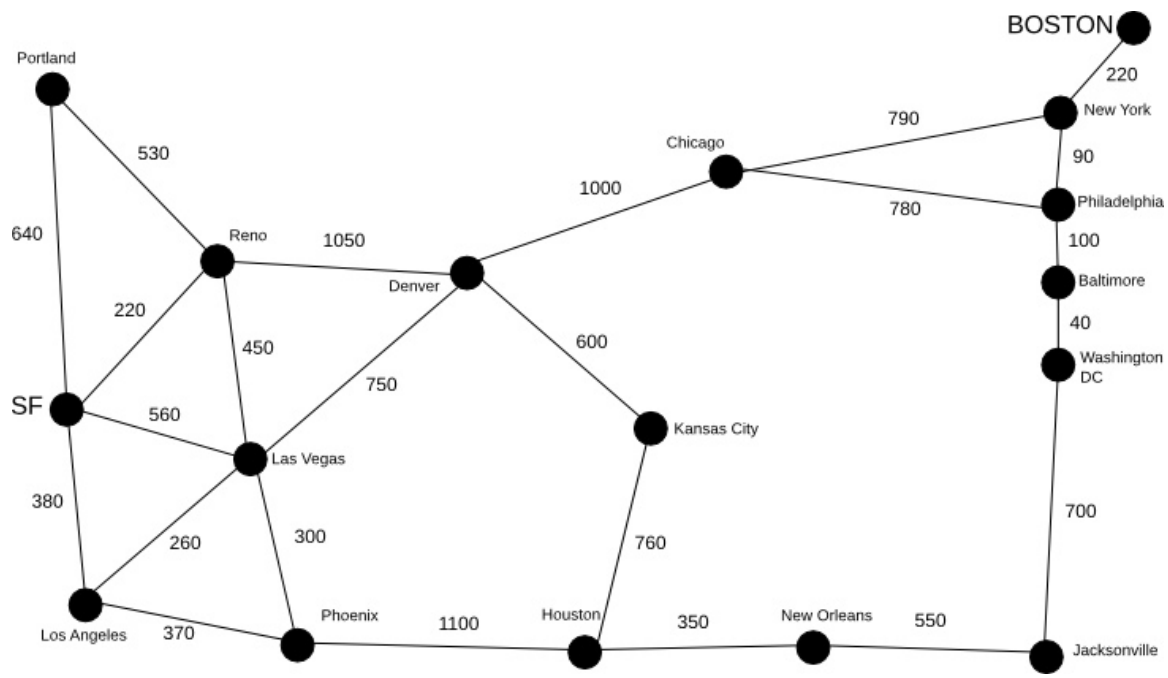
At first, not obvious how to do this.
But: such a numbering is just a Hamiltonian cycle in the n-cube.



Weighted Graphs

A weighted graph is a graph $G(V,E)$ together with a function
$$w : E \to [0, \infty)$$

For $e \in E$, the number $w(e)$ is the weight of e.



Weighted Graphs

Graph	Vertices	Edges	Weights
communication	computers	fiberoptic cables	response time
air travel	airports	flights	flight times
car travel	street corners	streets	distances
Kevin Bacon	actors	common movies	1
stock market	stocks	transactions (directed edges)	cost
operations research	projects	dependencies (directed edges)	times

Distance Problems

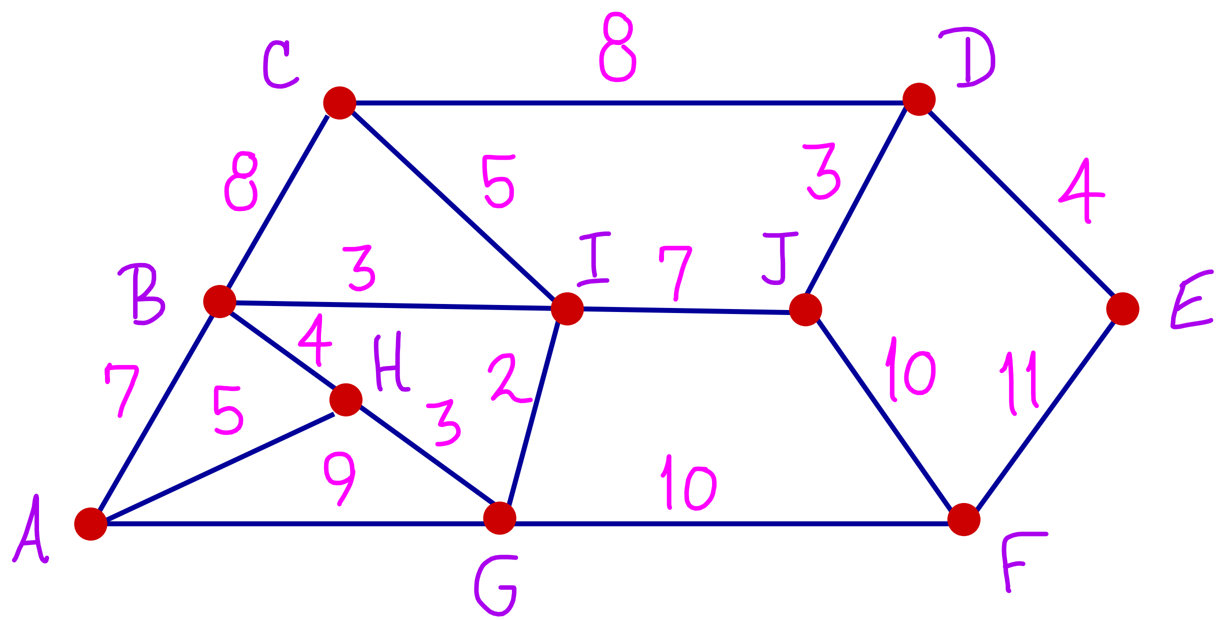
Traveling Salesman Problem. Given a list of cities to visit, what is the minimum distance you need to travel?

TSP is really a question about weighted graphs.

Easier Problem. Given two vertices in a weighted graph, what is their "distance."

EXAMPLE

PROBLEM. Find the distance between A and E.



How to find the shortest path in general?

Dijkstra's Algorithm

To find the distances from a given vertex A in a weighted graph to all other vertices, do the following.

First, give A the permanent label 0, and give all other vertices the temporary label ∞.

Then repeat the following step:

Edsger Dijkstra

Find the vertex v with the newest permanent label.

For each vertex v' adjacent to v with a temporary label, check if

label of v + w(vv') ≤ label of v'

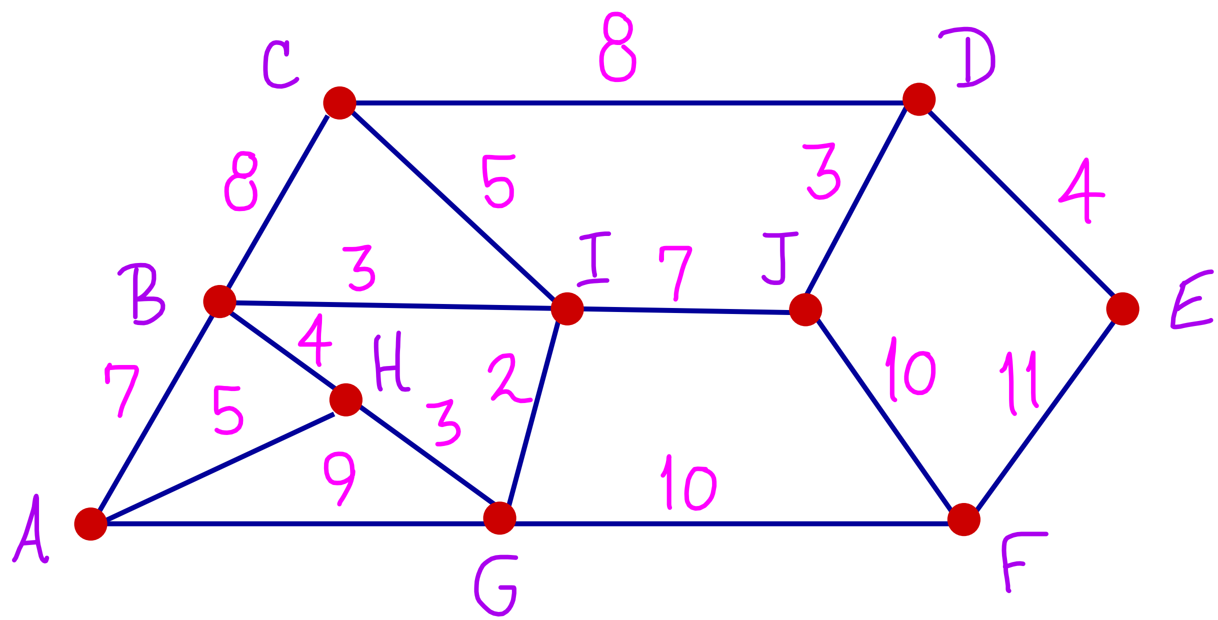
If so, change the temporary label of v'.

Make the smallest temporary label permanent.

Permanent labels are the distances from A.

Dijkstra's Algorithm

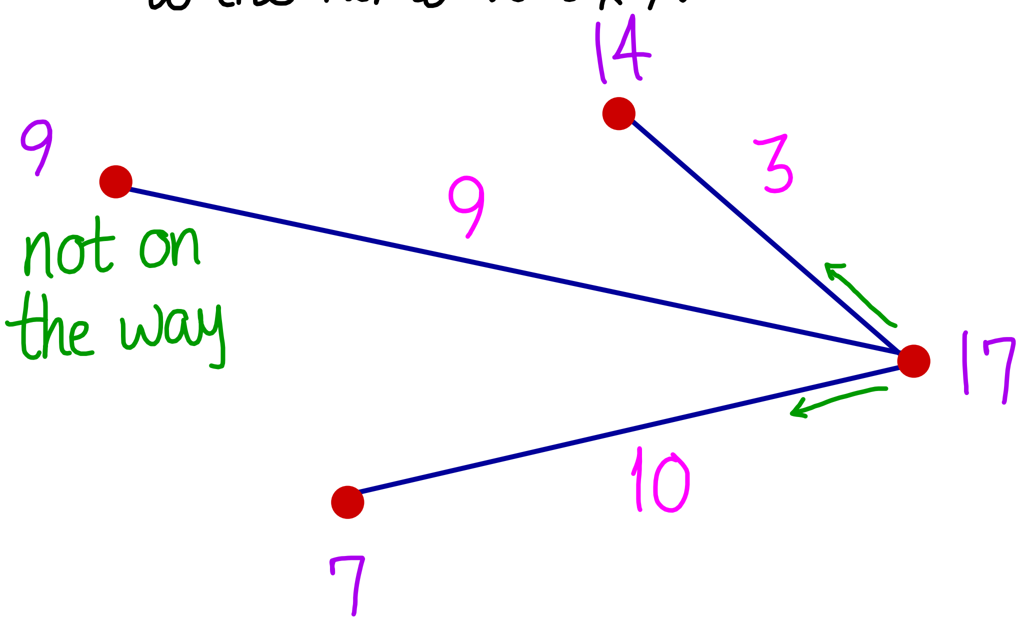
Find the distance from A to each other vertex.



Dijkstra's Algorithm

What if we further want to find a walk between two vertices with the shortest length (not just the distance between the two vertices)?

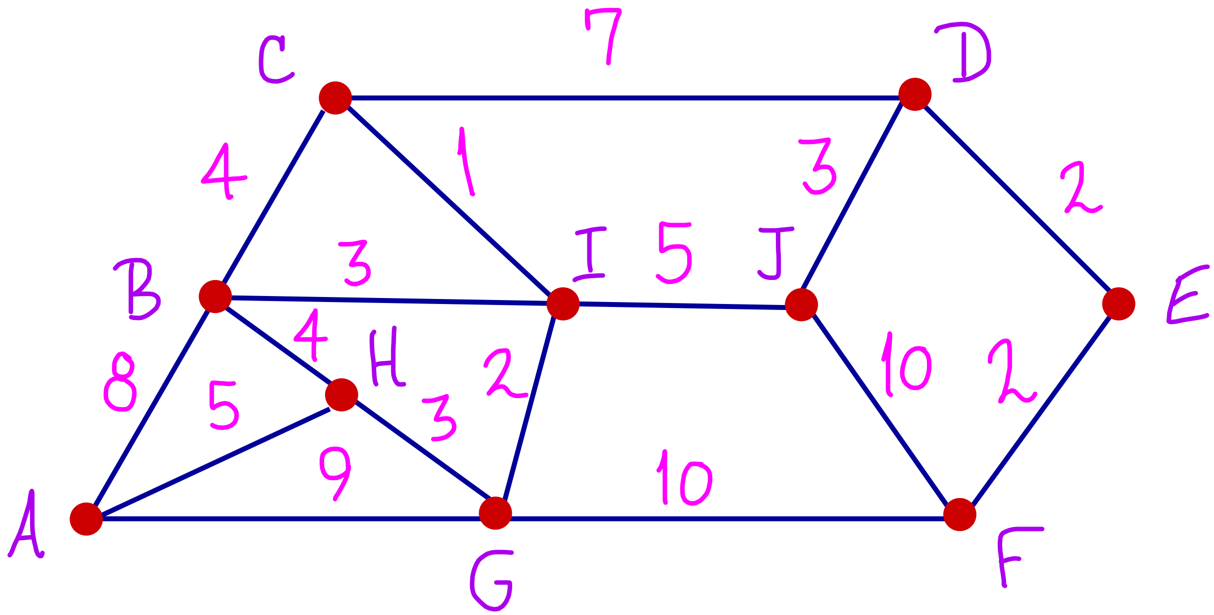
Idea: Every time we make a label permanent, draw a little arrow from that vertex to all other vertices that are "en route" to the home vertex A



Then, follow the arrows to find all shortest walks home.

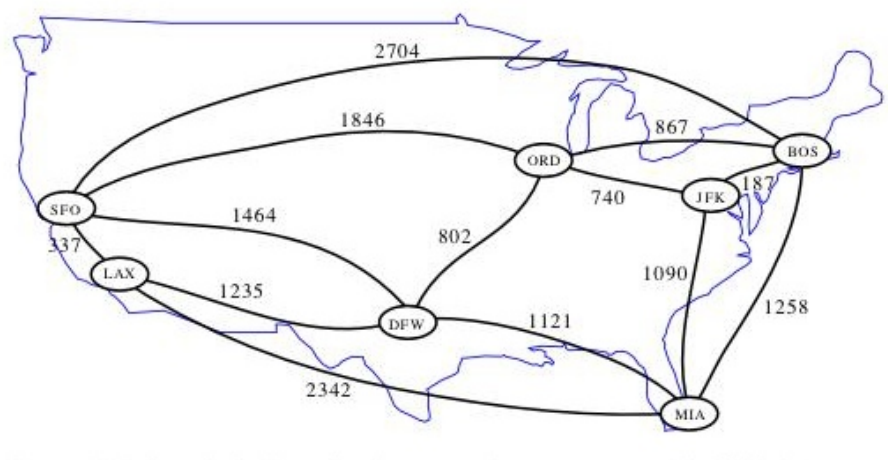
Dijkstra's Algorithm

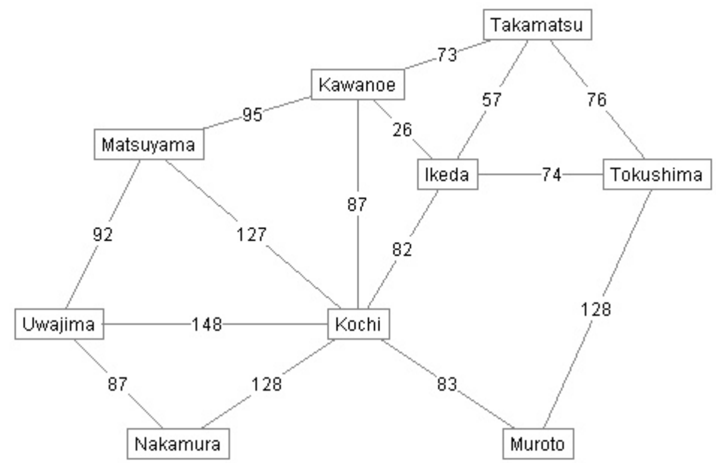
Find all shortest paths from A to E.



Dijkstra's Algorithm

Find the shortest paths…





from LAX to JFK

from Nakamura to Tokushima

DIJKSTRA'S ALGORITHM

What is the complexity of Dijkstra's algorithm, if size is measured in the number of vertices and cost is measured in terms of number of operations (=additions and comparisons)?

Floyd-Warshall Algorithm

Idea: Number the vertices $v_1, \ldots, v_n$.

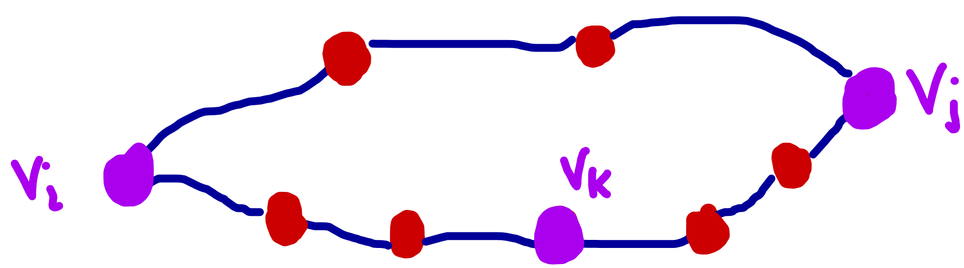
Step k: Find the shortest path from v_i to v_j if you are only allowed to use $v_1, \ldots, v_k$ as intermediate vertices (= pit stops).
Can write this info in a matrix M_k.
Write ∞ if there is no path.

Do this for $k = 0, \ldots, n$. (At Step 0, no pit stops allowed.)

The ij-entry of M_n is the distance from v_i to v_j.

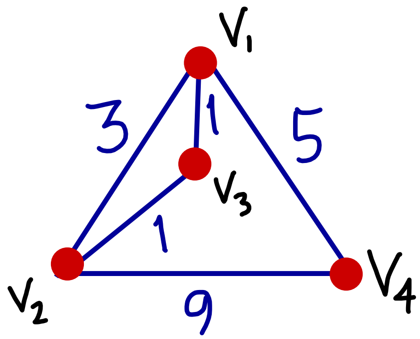
Key observation. For $k \geq 1$:

$$M_k(i,j) = \min\{M_{k-1}(i,j),\ M_{k-1}(i,k) + M_{k-1}(k,j)\}$$



Floyd-Warshall Algorithm

Example.



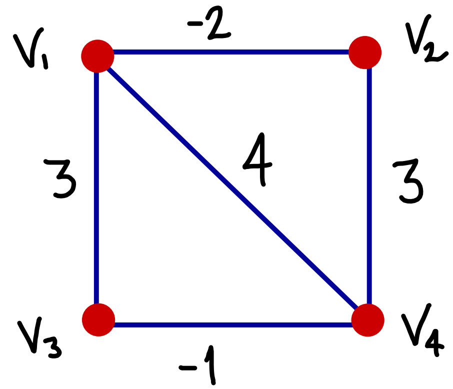
$$M_0 = \begin{pmatrix} 0 & 3 & 1 & 5 \\ & 0 & 1 & 9 \\ & & 0 & \infty \\ & & & 0 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0 & 3 & 1 & 5 \\ & 0 & 1 & 8 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 3 & 1 & 5 \\ & 0 & 1 & 8 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 2 & 1 & 5 \\ & 0 & 1 & 7 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix} \qquad M_4 = \begin{pmatrix} 0 & 2 & 1 & 5 \\ & 0 & 1 & 7 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix}$$

Note: M_k has same row/col k as M_{k-1}.

Floyd-Warshall Algorithm

Find all distances using the Floyd-Warshall algorithm.



Dijkstra vs Floyd-Warshall

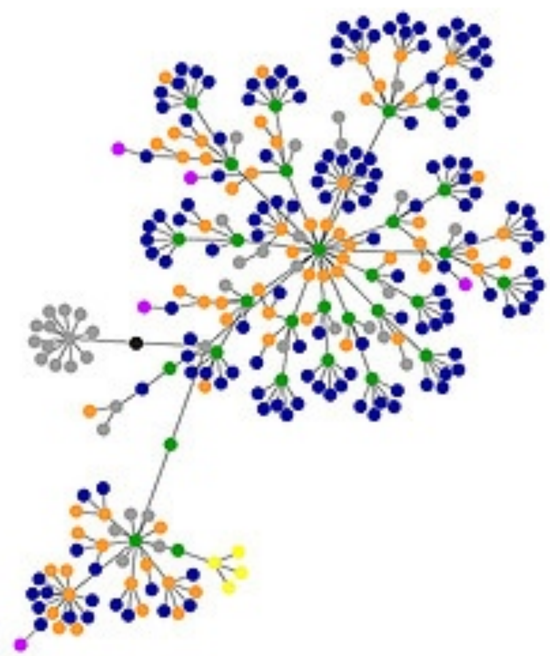
To find distances for all pairs of vertices, we need to run Dijkstra's algorithm n times $\rightsquigarrow \mathcal{O}(n^3)$.

Floyd-Warshall is also $\mathcal{O}(n^3)$, but is quicker for large graphs.

One advantage to Floyd-Warshall is that it even works with negative edge weights.

TREES

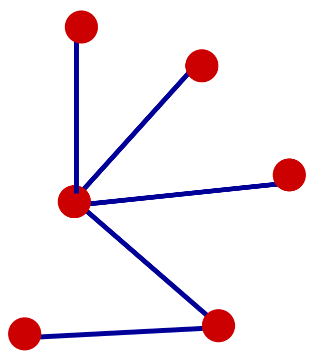
A tree is a connected graph with no circuits.



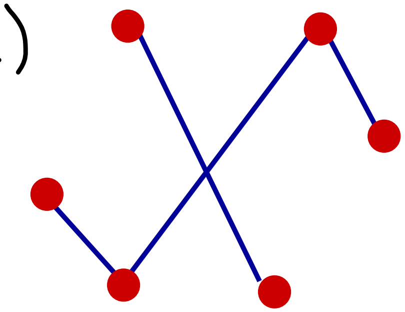
TREES

Which of the following graphs are trees?

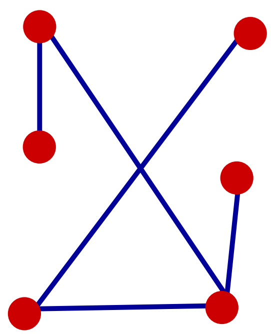
(i)



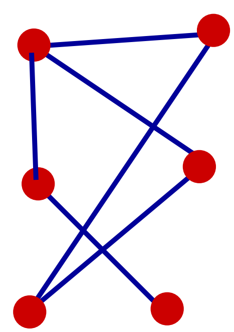
(ii)



(iii)



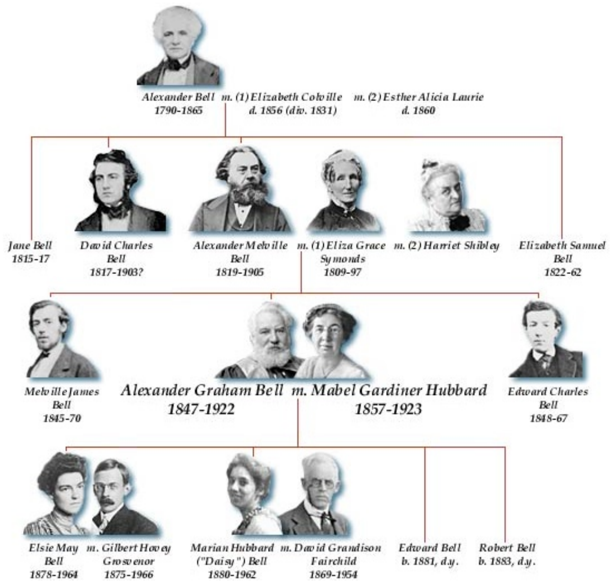
(iv)

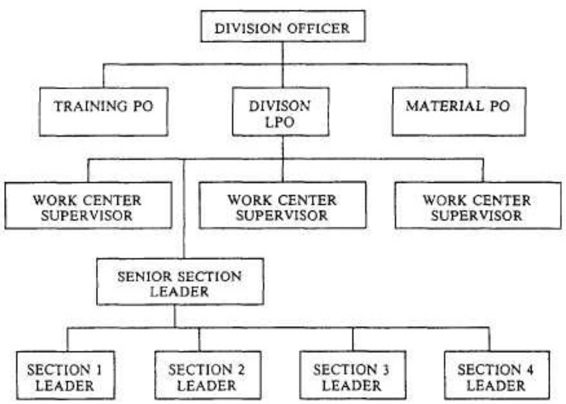


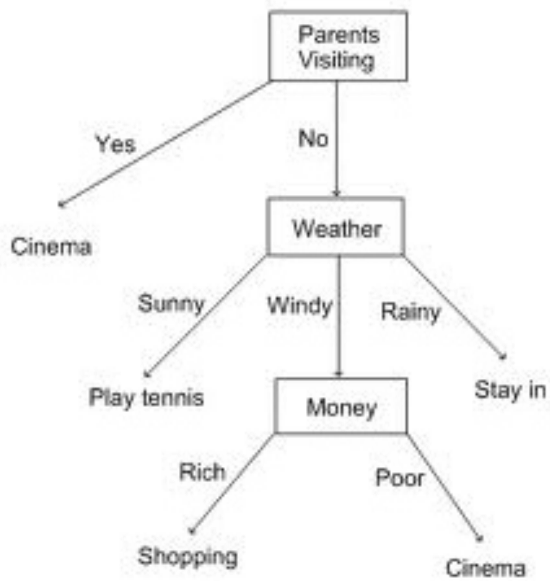
TREES

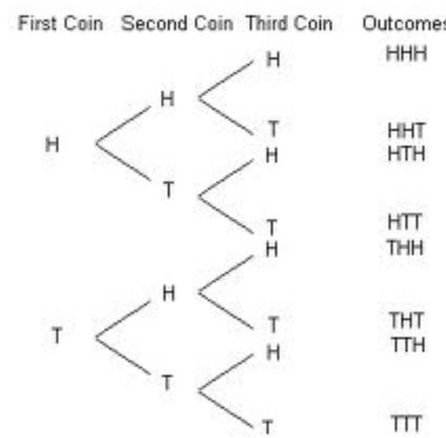
List all trees with 5 or fewer vertices up to isomorphism.

Applications of Trees









and many more...

Characterizing Trees

Theorem. Let G be a graph with n vertices. The following are equivalent:

(i) G is a tree (i.e. G is connected with no circuits)
(ii) G is connected and has no cycles.
(iii) G is connected and has $n-1$ edges.
(iv) Between any two vertices of G there is a unique walk that does not repeat any edges.

Also:

(v) G has $n-1$ edges and no cycles
(vi) G is connected, but removing any edge makes it disconnected.
(vii) G has no cycles, but adding any edge creates one.

etc...

Application To Chemistry

A hydrocarbon has the form C_nH_{2n+2}. Carbon has degree 4 and Hydrogen has degree 1.

Problem. Find all hydrocarbons for $n = 1, 2, 3, 4$.

Characterizing Trees

Theorem. Let G be a graph with n vertices. The following are equivalent:

(i) G is a tree (i.e. G is connected with no circuits)
(ii) G is connected and has no cycles.
(iii) G is connected and has $n-1$ edges.
(iv) Between any two vertices of G there is a unique walk that does not repeat any edges.

12.2 Spanning Trees

SPANNING TREES

A spanning tree for a graph G is a subgraph that is a tree and that contains every vertex.

A minimal spanning tree for a weighted graph is a spanning tree of least weight.

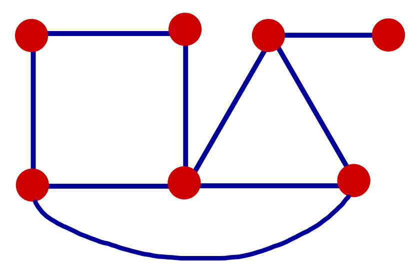
Application: Given a network of roads, which roads should you pave so that (a) all towns are connected and (b) we use the least amount of asphalt?

Spanning Trees

How to find a spanning tree?

One answer: Delete all edges until there are no cycles.

Example. How many spanning trees can you find?

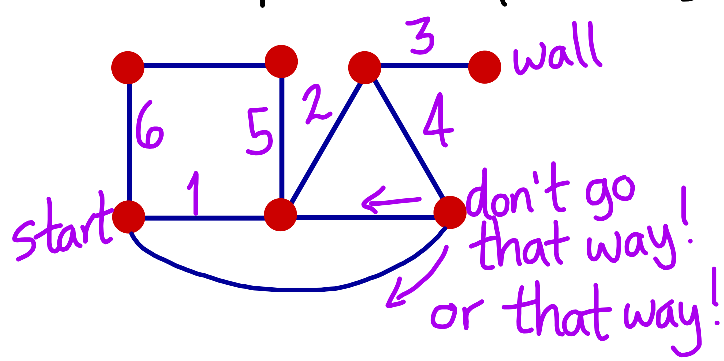


Question. How to find all spanning trees? How many are there?

Could hunt for cycles, delete edges. Inefficient!

Depth-First Search and Breadth-First Search

Depth-first: Start at some point in the graph.
Draw a long path, go as far as possible.
When you hit a wall (= degree 1 vertex),
or an edge that creates a cycle with your
path, back up one step and go in a new direction.



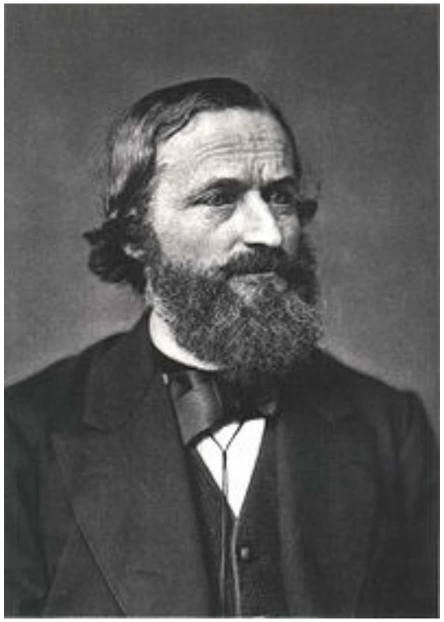
Breadth-first: Use as many edges from start point as
possible. Then from the endpoints of all
those edges use as many edges as possible, etc.

Kirchhoff's Theorem

Given a graph with vertices $v_1, \ldots, v_n$, make a matrix M with (i,i)-entry the degree of v_i and all other (i,j)-entries given by: -1 if v_iv_j is an edge
0 otherwise

THEOREM. Given a graph G, make the matrix M as above. Delete the i^{th} row and the j^{th} column to obtain a matrix M'. Then:

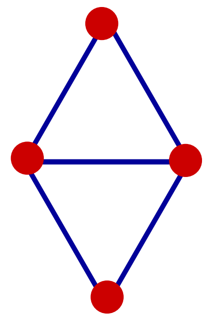
$$(-1)^{i+j} \det(M') = \#\text{ spanning trees for } G.$$



Gustav Kirchhoff

Kirchhoff's Theorem

Example.



12.3 Minimal Spanning Tree Algorithms

Krushkal's Algorithm

Goal: Find a minimal spanning tree for a given graph.

Want something more efficient than enumerating all trees.

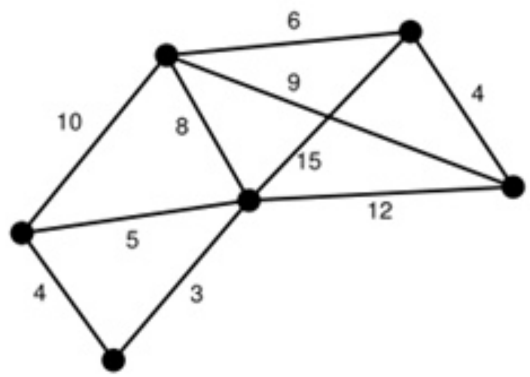
The Algorithm. Set $T = \emptyset$.
Consider all edges e so $T \cup \{e\}$ has no circuits.
Choose the edge e of smallest weight with this property.
Replace T with $T \cup \{e\}$.
Repeat until T is a spanning tree.

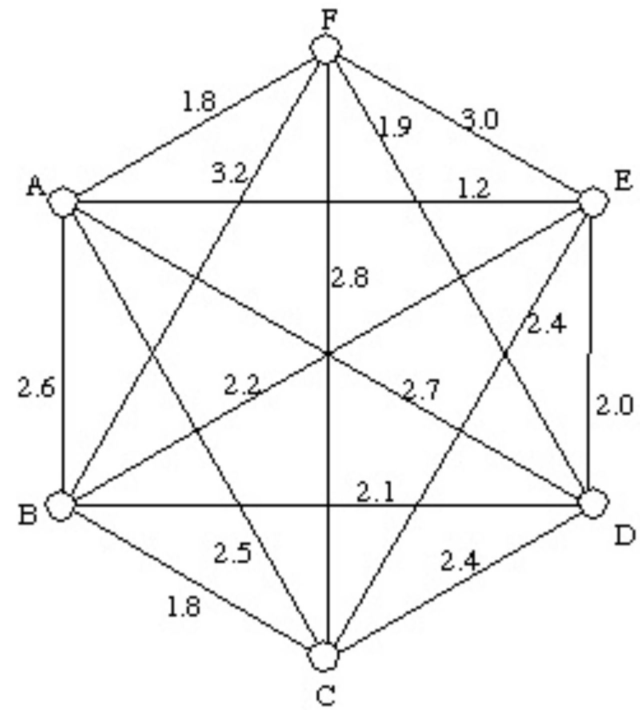
Note: The number of steps is one less than the # of vertices.

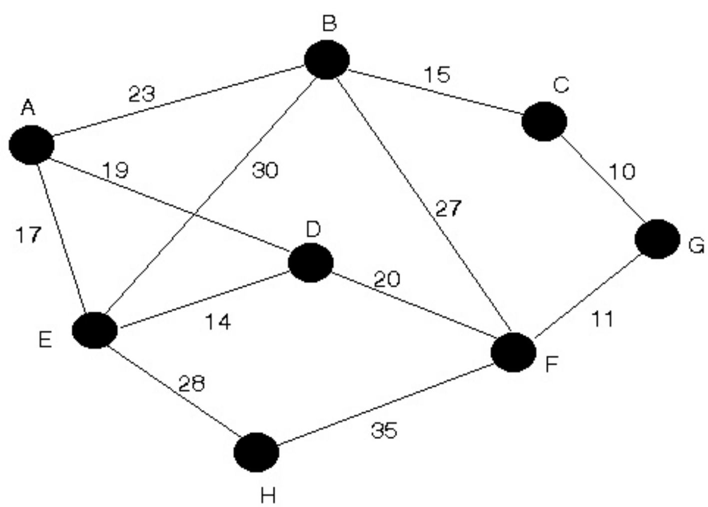
Krushkal's algorithm is an example of a "greedy algorithm"

Krushkal's Algorithm

Find minimal spanning trees for the following weighted graphs.







KRUSHKAL'S ALGORITHM

Why does the algorithm work?

Let $e_1, \ldots, e_{n-1}$ be the edges chosen by Krushkal's algorithm, in order.

Prove the following statement by induction:

$\{e_1, \ldots, e_k\}$ is contained in some minimal spanning tree.

Base case: $k=0$, i.e. $\emptyset$ contained in some minimal spanning tree. ✓

Suppose $\{e_1, \ldots, e_k\}$ contained in some minimal spanning tree T, but e_{k+1} is not in T. $\rightsquigarrow$ $T \cup e_{k+1}$ has a cycle.
There is an edge f contained in this cycle that is not equal to $e_1, \ldots, e_{k+1}$ (the e_i form a tree, so they form no cycles).
Now, f and e_{k+1} have same weight, otherwise weight of $T - f + e_{k+1}$ is less than weight of T. We see $T - f + e_{k+1}$ is the desired tree. $\blacksquare$

Prim's Algorithm

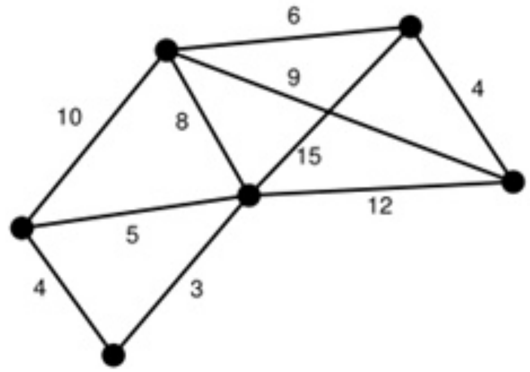
Idea: Grow a tree from a vertex.

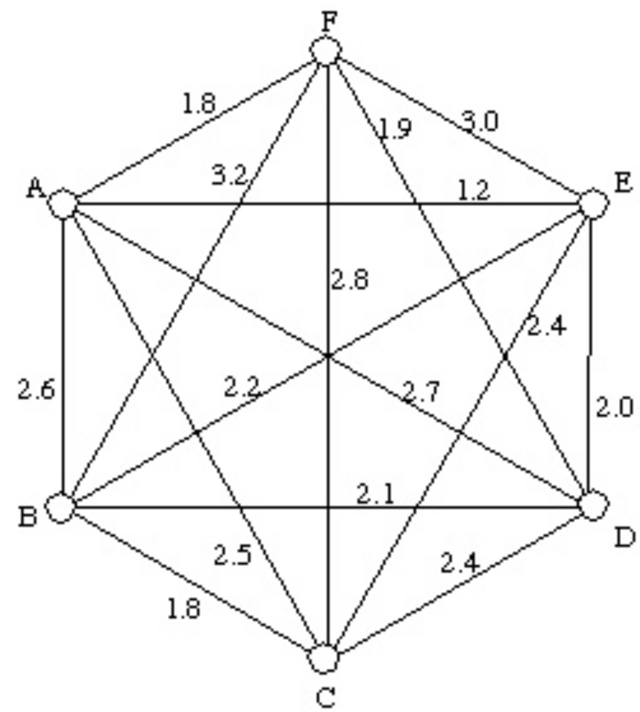
The algorithm. Set $T = v$ (any vertex)
Choose an edge e of minimal weight so $T \cup \{e\}$ is a tree
Replace T with $T \cup \{e\}$.
Repeat until T is a spanning tree.

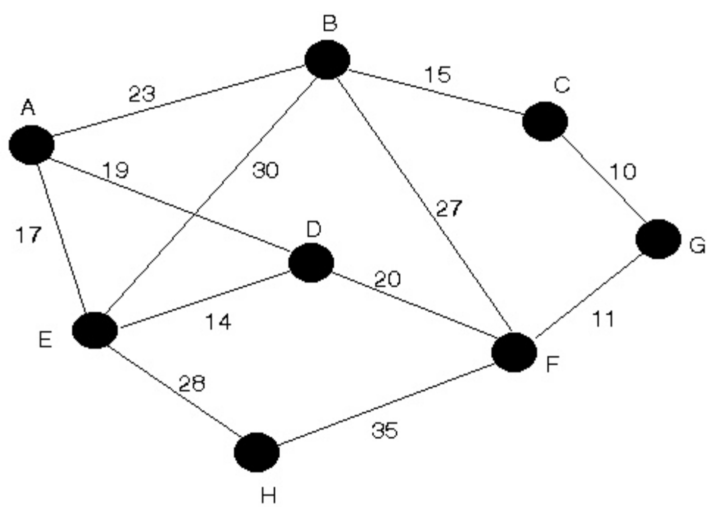
Note: We know $T \cup \{e\}$ is a tree if $T \cap e$ is a single vertex.

Prim's Algorithm

Find minimal spanning trees for the following weighted graphs.







KRUSHKAL'S ALGORITHM VS. PRIM'S ALGORITHM

What is the complexity? size = # edges
cost = # comparisons

KRUSHKAL: $O(n \log n + n^2)$

PRIM: $O(n^2)$

Check these! Idea: order the remaining edges. Then, need to check which can be added to the current tree by comparing the endpoints of each edge with the vertices of the current tree.

The advantage over Krushkal's algorithm is that there are fewer edges to check at each step. In fact, Prim is $O(n^2)$.

Spanning Trees

A spanning tree for a graph G is a subgraph that is a tree and that contains every vertex.

A minimal spanning tree for a weighted graph is a spanning tree of least weight.

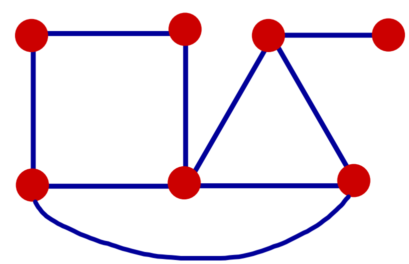
Application: Given a network of roads, which roads should you pave so that (a) all towns are connected and (b) we use the least amount of asphalt?

Spanning Trees

How to find a spanning tree?

One answer: Delete all edges until there are no cycles.

Example. How many spanning trees can you find?

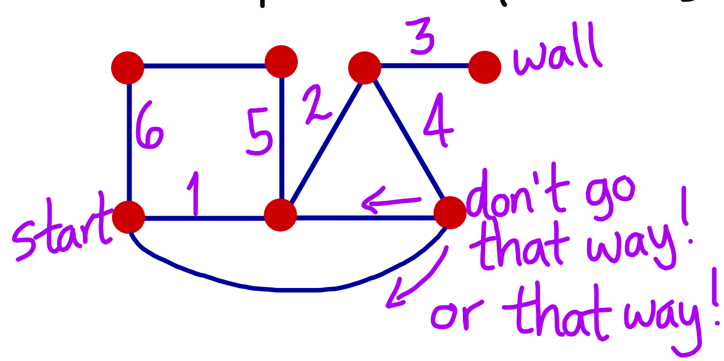


Question. How to find all spanning trees? How many are there?

Could hunt for cycles, delete edges. Inefficient!

Depth-First Search and Breadth-First Search

Depth-first: Start at some point in the graph.
Draw a long path, go as far as possible.
When you hit a wall (= degree 1 vertex),
or an edge that creates a cycle with your
path, back up one step and go in a new direction.



Breadth-first: Use as many edges from start point as
possible. Then from the endpoints of all
those edges use as many edges as possible, etc.

MAZES

One algorithm for solving a maze is to put your right hand on the wall and walk.

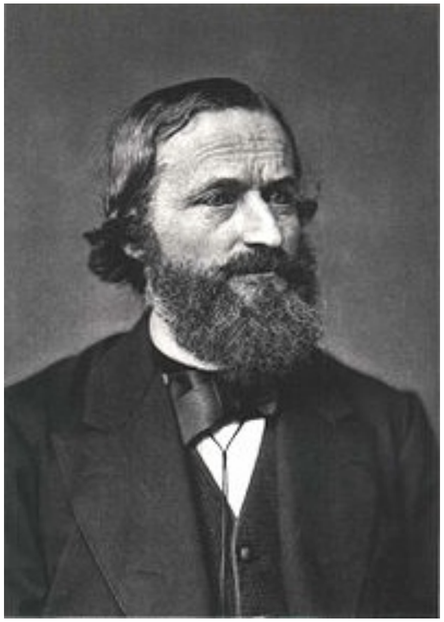
Is this a depth-first or breadth-first algorithm?

Kirchhoff's Theorem

Given a graph with vertices $v_1, \ldots, v_n$, make a matrix M with (i,i)-entry the degree of v_i and all other (i,j)-entries given by: -1 if v_iv_j is an edge
0 otherwise

Theorem. Given a graph G, make the matrix M as above. Delete the i^{th} row and the j^{th} column to obtain a matrix M'. Then:

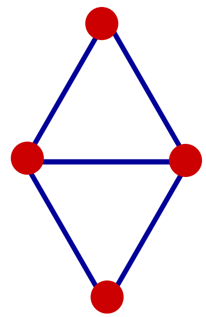
$$(-1)^{i+j} \det(M') = \#\text{ spanning trees for } G.$$



Gustav Kirchhoff

Kirchhoff's Theorem

Example.



Krushkal's Algorithm

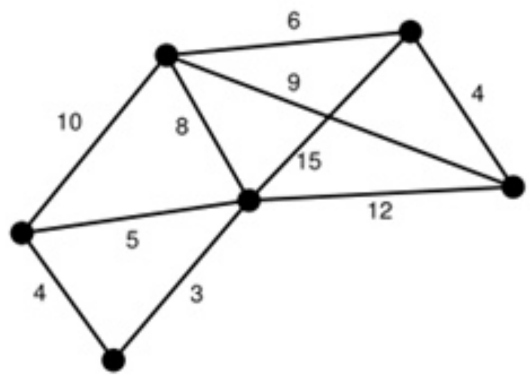
The Algorithm. Set $T = \emptyset$.
Consider all edges e so $T \cup \{e\}$ has no circuits.
Choose the edge e of smallest weight with this property.
Replace T with $T \cup \{e\}$.
Repeat until T is a spanning tree.

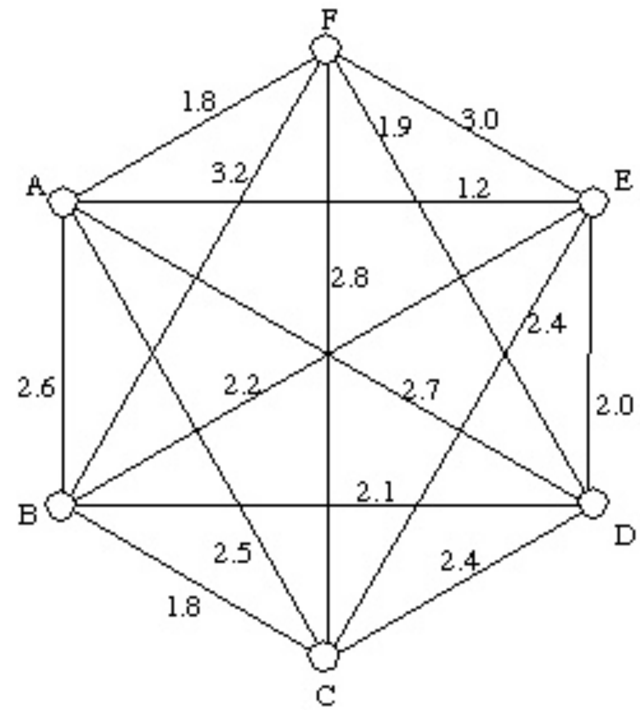
Note: The number of steps is one less than the # of vertices.

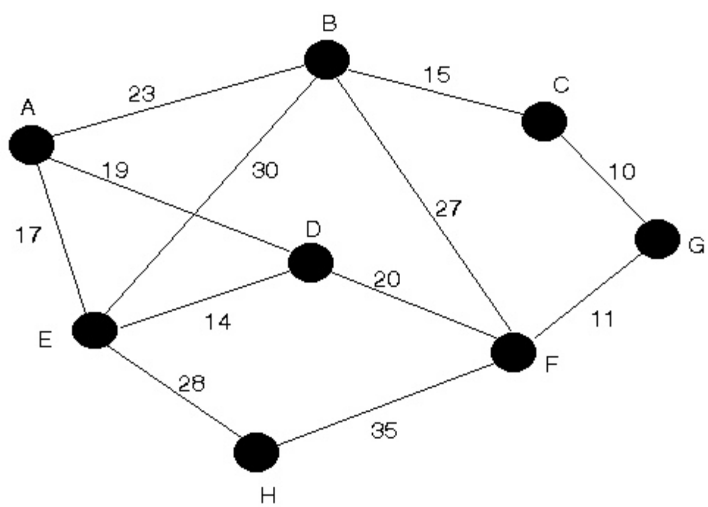
Krushkal's algorithm is an example of a "greedy algorithm"

Krushkal's Algorithm

Find minimal spanning trees for the following weighted graphs.







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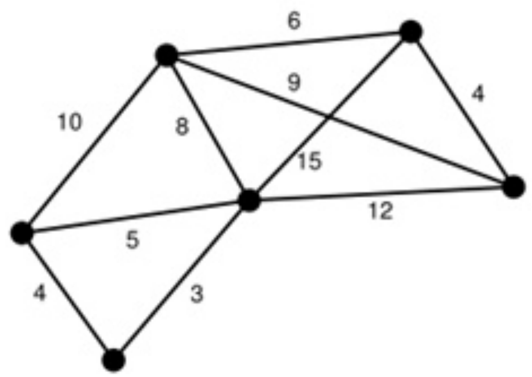
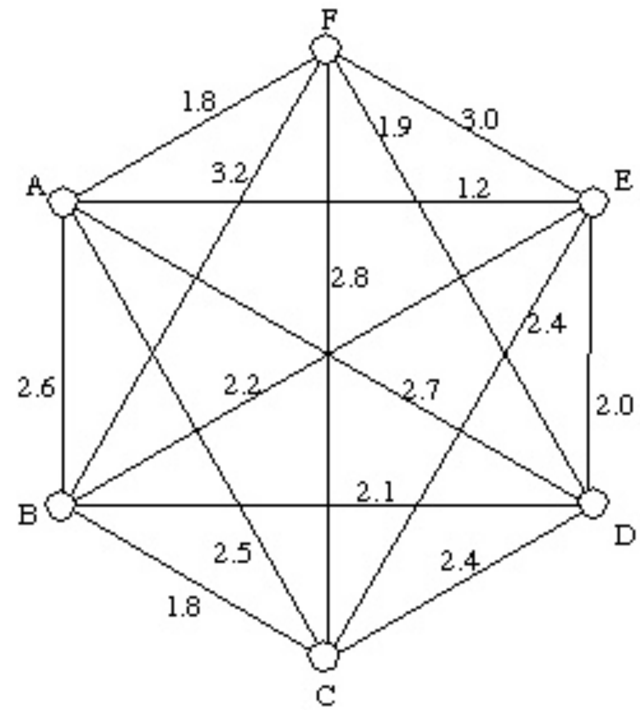
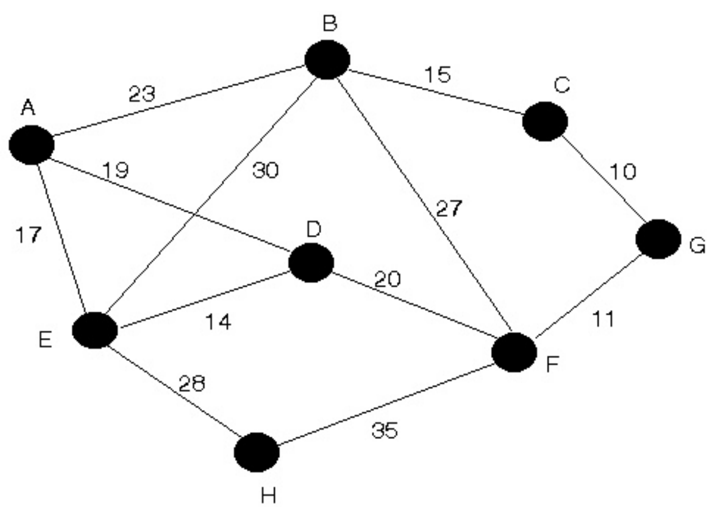






KRUSHKAL'S ALGORITHM VS. PRIM'S ALGORITHM

What is the complexity?

size = # edges
cost = # comparisons

KRUSHKAL: $O(n \log n + n^2)$

PRIM: $O(n^2)$

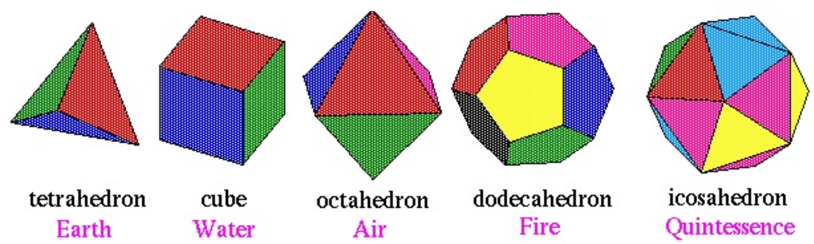
Planar Graphs

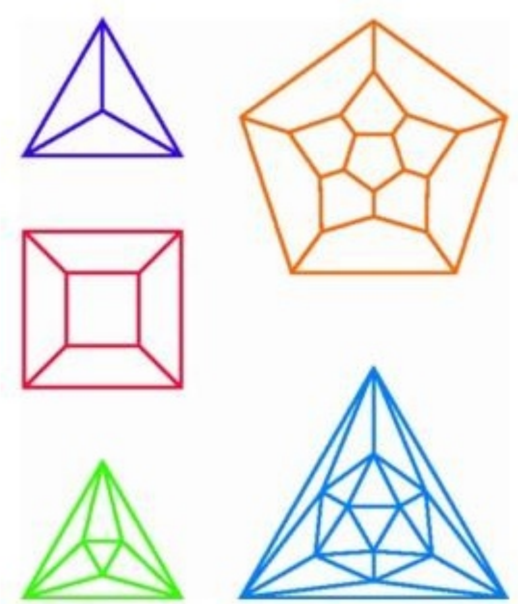
A graph is *planar* if it can be drawn in the plane so that no two edges cross.

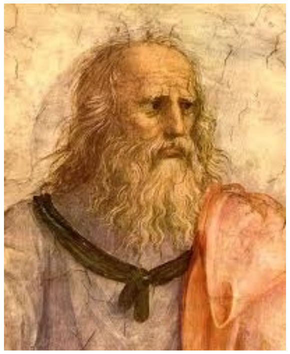
The Three House - Three Utility Problem asks whether or not $K_{3,3}$ is planar.

Platonic Solids

One collection of interesting planar graphs comes from the five Platonic solids:



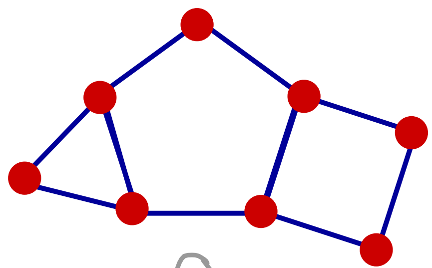




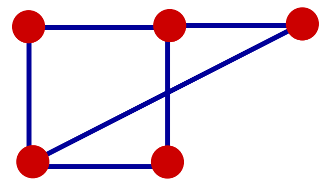
Plato

Planar Graphs

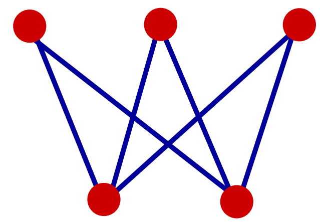
Which of the following graphs are planar?



1



2



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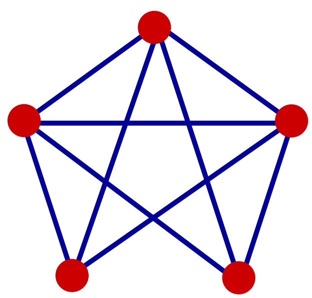
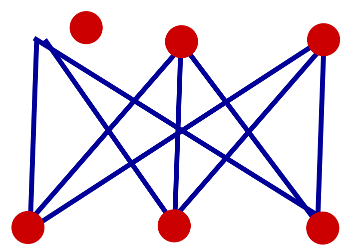


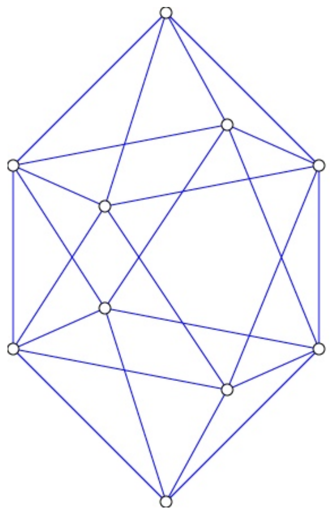
4



5

Planar Graphs

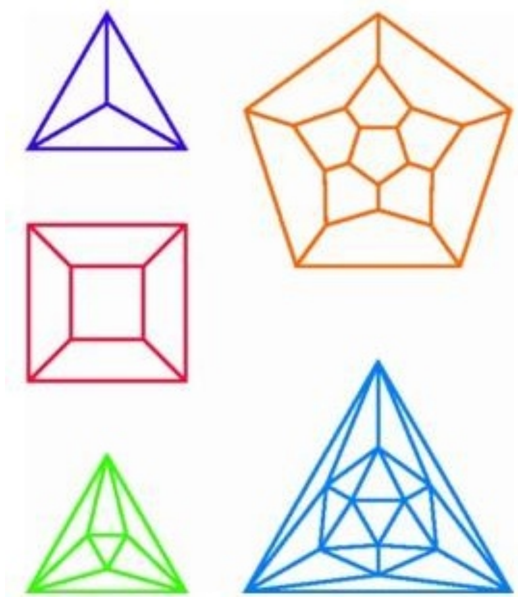
Is this graph planar?



Vertices, Edges, and Faces

A planar drawing of a planar graph divides the plane into distinct regions, or faces.

	vertices	edges	faces
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			



What is the pattern?

Euler's Theorem

Theorem. Any planar drawing of a graph with V vertices, E edges, and F faces satisfies

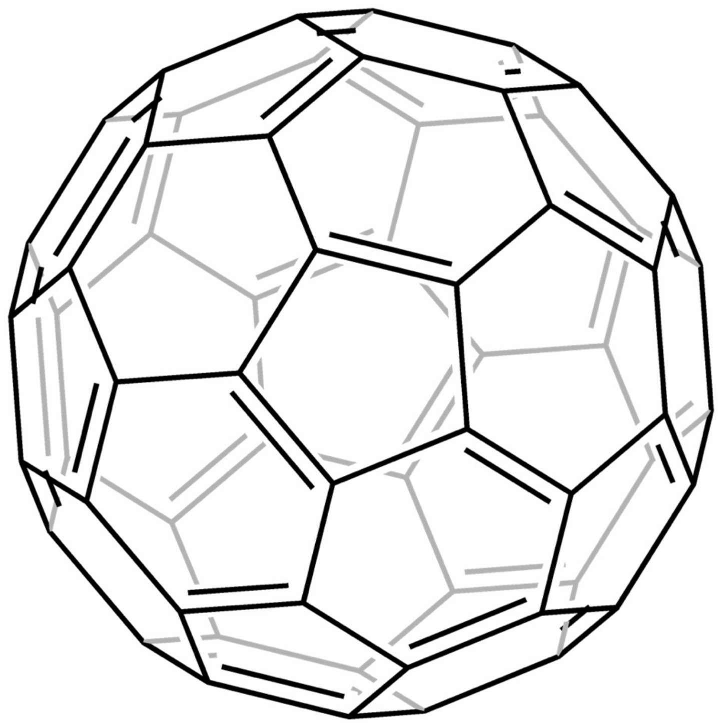
$$V - E + F = 2$$

In 1988, the Mathematical Intelligencer ran a survey. It was decided that the 5 most beautiful results in mathematics were:

(i) Euler's identity $e^{ix} = \cos x + i \sin x$
(ii) Euler's polyhedral formula $V - E + F = 2$
(iii) Euclid's proof of the infinitude of the primes
(iv) Euclid's proof that there are only 5 regular solids
(v) Euler's summation $\sum 1/n^2 = \pi^2/6$

Euler's Theorem

Does the Buckeyball satisfy $V - E + F = 2$?



Euler's Theorem

THEOREM. Any planar drawing of a connected graph with V vertices, E edges, and F faces satisfies

$$V - E + F = 2$$

$K_{3,3}$ is not Planar

Theorem. $K_{3,3}$ is not planar.

K_5 IS NOT PLANAR

THEOREM. If a planar graph has V vertices and E edges, then $E \leq 3V - 6$.

COROLLARY. K_5 is not planar.

Degrees

THEOREM. Every planar graph has at least one vertex whose degree is less than 6.

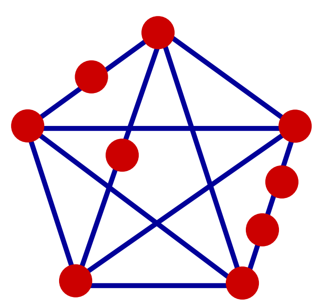
More Nonplanar Graphs

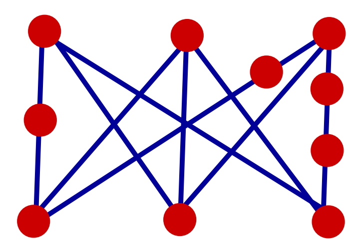
So far, we know K_5 and K_{33} are not planar.
It follows that K_n is not planar for $n \geq 5$,
$K_{m,n}$ is not planar for $m, n \geq 3$.

More generally:

Proposition. Any graph that contains K_5 or $K_{3,3}$ as a subgraph is not planar.

Note also any subdivision of K_5 or K_3 is nonplanar:





Proposition. Any graph that contains a subdivision of K_5 or $K_{3,3}$ as a subgraph is not planar.

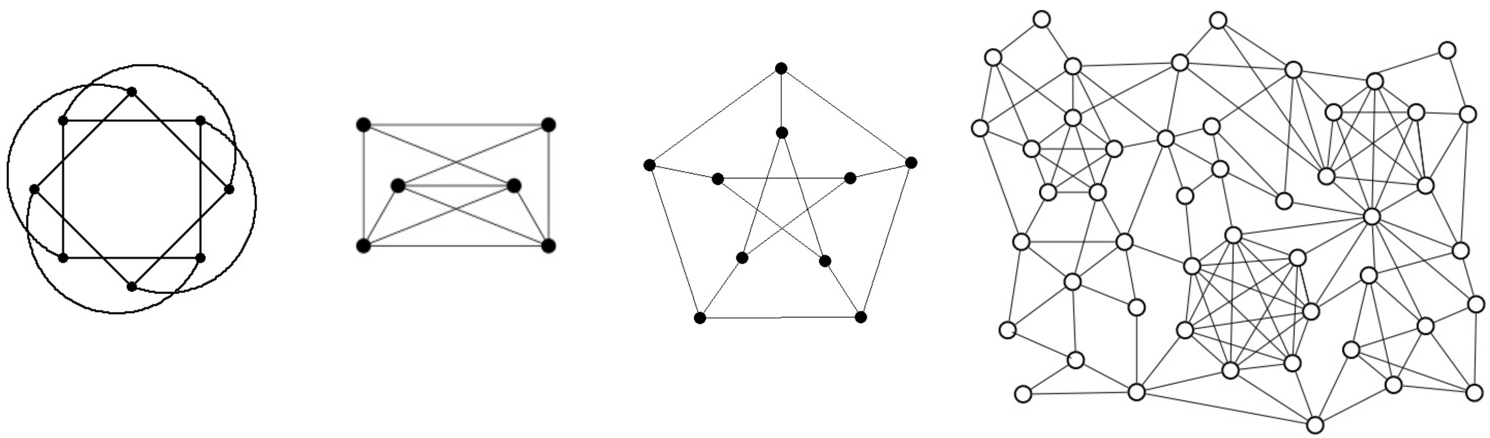
Kuratowski's Theorem

Amazingly, the converse is also true:

Theorem. A graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.

Proof. See web site.

Which of the following graphs are planar?



Platonic Solids

A Platonic solid is a 3-dimensional solid with polygonal faces, and satisfying:

(i) The faces are regular and congruent.

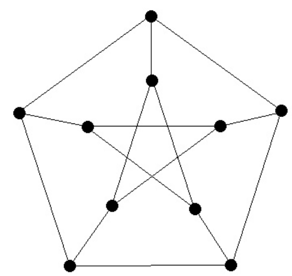
(ii) The same number of faces meet at each vertex.

(iii) The line connecting any two points on the solid is contained in the solid.

WAGNER'S THEOREM

A graph H is a minor of a graph G if H is obtained from G by taking a subgraph and collapsing some edges.

THEOREM. A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.



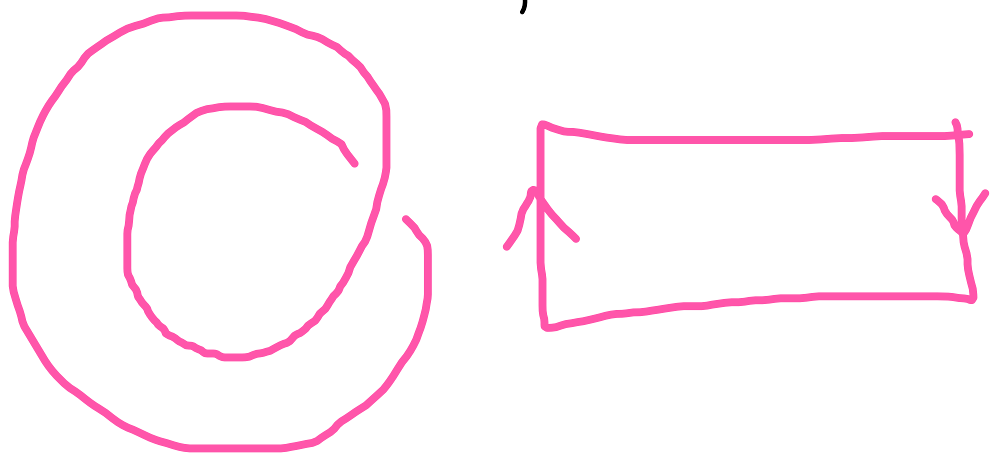
FÀRY'S THEOREM

THEOREM. Every planar graph can be drawn in the plane using only straight lines.

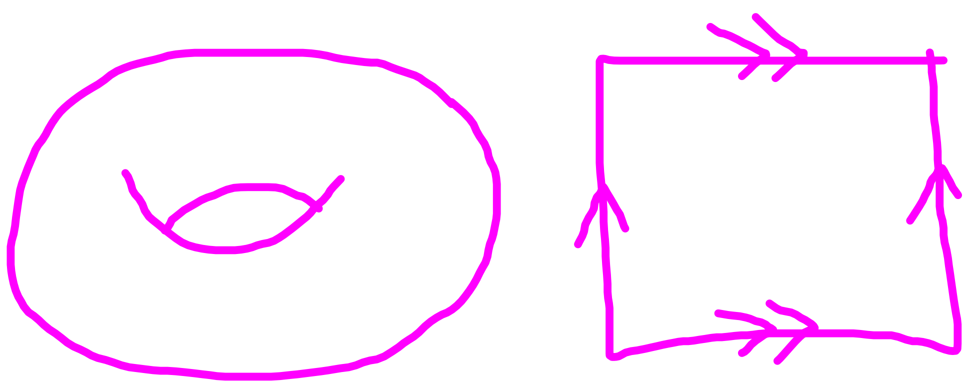
The proof uses the art gallery theorem...

Other Surfaces

What are the largest m, n so K_n and $K_{m,n}$ can be drawn without crossings on a Möbius strip



or a torus?



13.2 Coloring Graphs

THE FOUR COLOR PROBLEM

Show that, given any map in the plane, you can color it with four colors so that adjacent regions have different colors.

Notes. (i) Each region must be a connected "blob".
(ii) "Adjacent" means the regions meet in a segment (not just a corner).

Why are these caveats needed?

Francis Guthrie

Is there a map that really requires 4 colors?

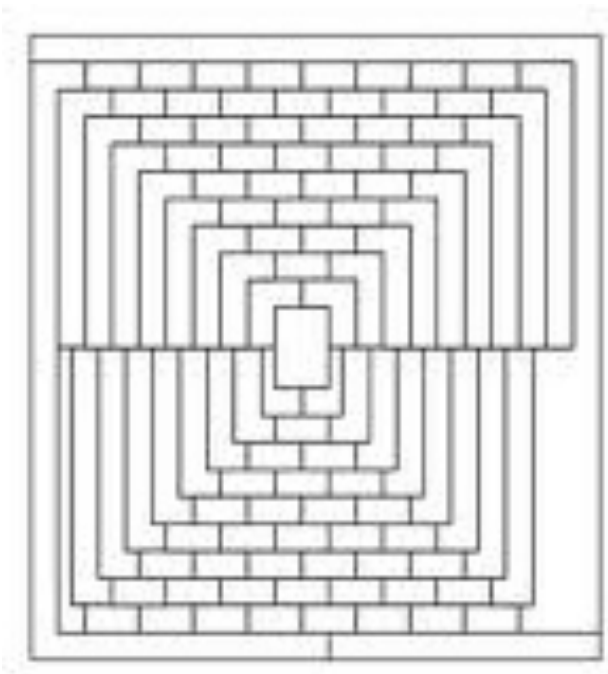
The Four Color Problem

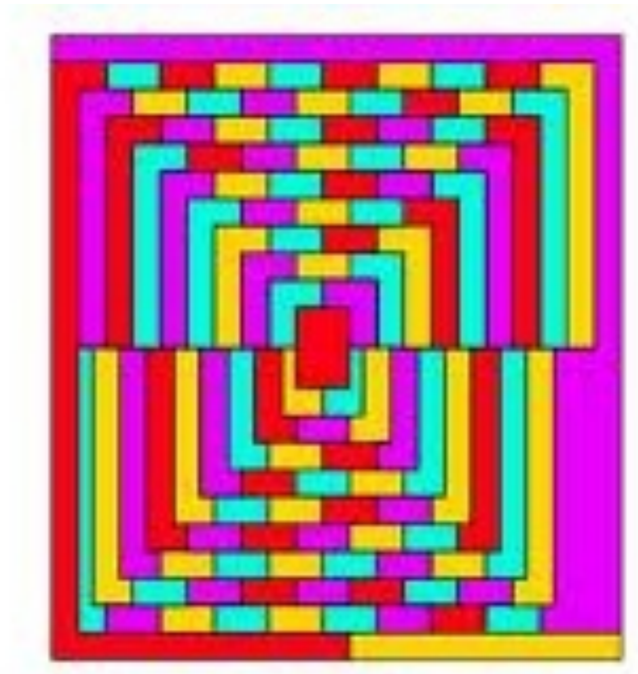
How many colors are needed?

Hint: Look at Nevada.

The Four Color Problem

How many colors are needed?





For more challenges: nikoli.com

The Four Color Problem

First posed in 1852 by Guthrie. Many tried to solve it.
Alfred Kempe (1879) and Pether Guthrie Tait (1880) both gave solutions that stood for 11 years.

Lewis Carroll wrote about it:

> "A is to draw a fictitious map divided into counties.
>
> B is to color it (or rather mark the counties with names of colours) using as few colours as possible.
>
> Two adjacent counties must have different colours.
>
> A's object is to force B to use as many colours as possible. How many can he force B to use?"

The problem was solved in 1976 by Appel and Haken. It was the first major theorem proven in large part by computer.

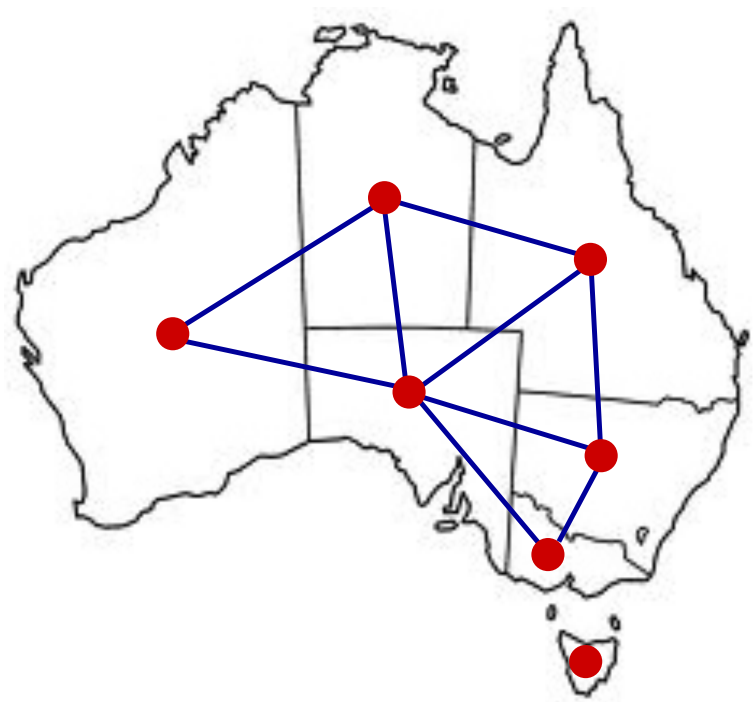
The proof has recently been simplified by Robin Thomas (GaTech) and his collaborators (still using computers).

Back to Graphs

Given a map, we get a graph $G(V,E)$ where

$V = \{\text{regions}\}$

$E = \{\text{pairs of adjacent regions}\}$



If the map is planar, then the graph is planar.

Coloring the map corresponds to coloring the vertices of the graph so that adjacent vertices have different colors.

Graph Coloring

A coloring of a graph is an assignment of colors to each of the vertices so that adjacent vertices have different colors.

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed for a coloring of G.

FACT. $1 \leq \chi(G) \leq |V|$

FACT. If G is isomorphic to H, then $\chi(G) = \chi(G)$.

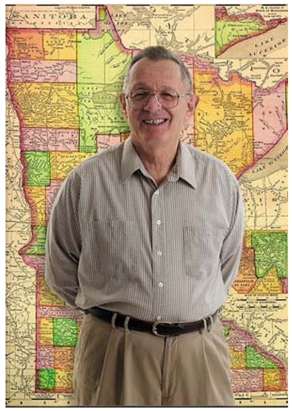
FACT. $\chi(K_n) = n$, $\chi(K_{m,n}) = 2$, and $\chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd.} \end{cases}$

FACT. If H is a subgraph of G then $\chi(H) \leq \chi(G)$

FACT. If G has a coloring with n colors, then $\chi(G) \leq n$.

The Four Color Theorem

Theorem. If G is planar, then $\chi(G) \leq 4$.



Kenneth Appel

Wolfgang Haken

Note: There is still no polynomial time algorithm for finding a coloring with 4 colors.

Applications

1. Sudoku. A vertex for each little square.
   An edge for two squares in same row, col, or 3×3 sqr.

2. Radio Frequencies. A vertex for each radio station.
   An edge between stations that are near eachother.

3. Scheduling. Example: Say there are 10 students taking

   1. Physics, Math, IE
   2. Physics, Econ, Geology
   3. Geology, Business
   4. Stat, Econ
   5. Math, Business
   6. Physics, Geology
   7. Business, Stat
   8. Math, Geology
   9. Physics, Comp Sci, Stat
   10. Physics, Econ, Comp Sci

What is the minimum number of final exam periods needed?

Six Colors Suffice

Proposition. If G is a planar graph then $\chi(G) \leq 6$.

Degrees and Colors

Proposition. For any graph G:

$$\chi(G) \leq (\text{largest degree of a vertex of } G) + 1$$

Proof. Same as above.

Computing χ

To show that $\chi(G) = n$, we generally have to show two things:

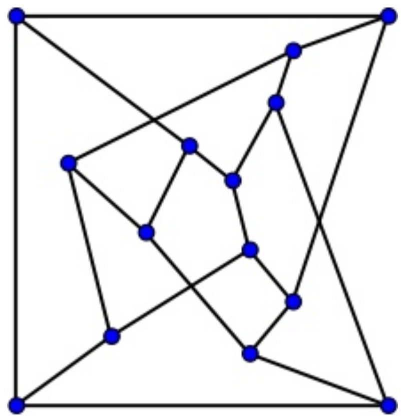
1. $\chi(G) \le n$

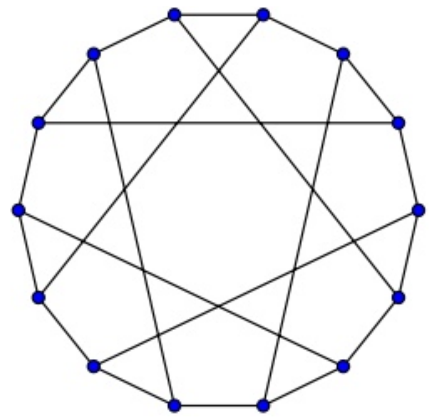
   Some possible reasons:
   - G has n vertices
   - G is bipartite
   - G is planar
   - Largest vertex degree is $n+1$
   - We know an explicit coloring with n vertices.

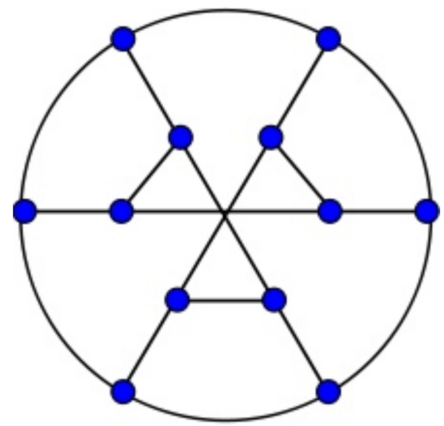
2. $\chi(G) \geq n$

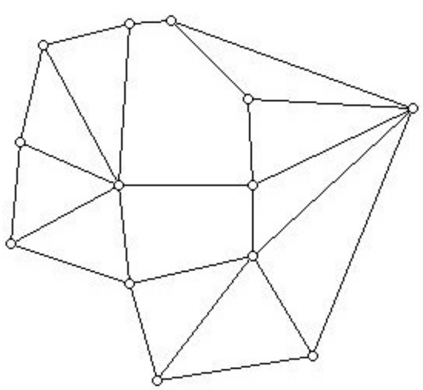
   Some possible reasons:
   - G contains H and $\chi(H) = n$
   - G contains H with $\chi(H) = n-1$ and a vertex adjacent to each vertex of H (cf. Nevada)

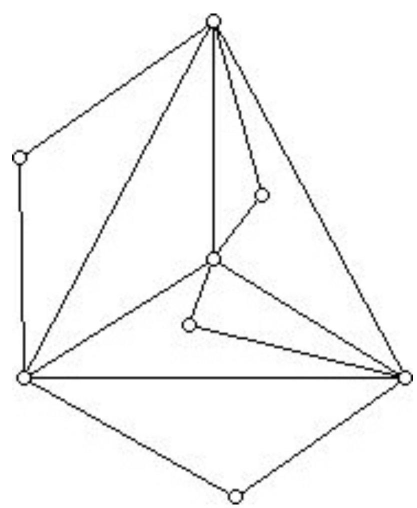
More Coloring Problems

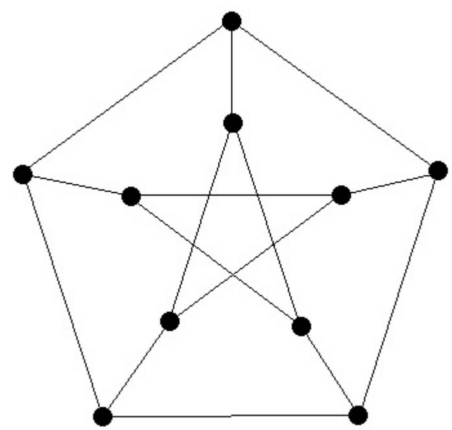












SCHEDULING

There are 10 students in the following classes:

Physics: Annie, Bob, Florence, Ingrid, Joe
Math: Annie, Elsa, Howard
Engineering: Annie
Geology: Bob, Cameron, Florence, Howard
Economics: Bob, Dylan, Joe
Business: Cameron, Elsa, Gordon
Statistics: Dylan, Gordon, Ingrid
Basket Weaving: Ingrid, Joe

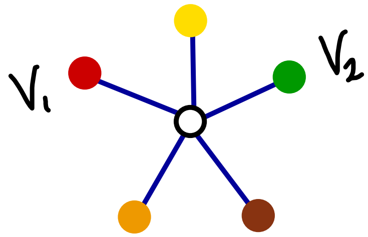
What is the minimum number of final exam periods needed?

Five Colors Suffice


Percy Heawood

Theorem. If G is a planar graph, then $\chi(G) \le 5$.

Proof. Induction on # vertices again.
Say G is a planar graph with n vertices.
As before, delete a vertex v of degree ≤ 5. Color $G-v$ with 5 colors. Can we reinsert v?



Case 1. There is no path from v_1 to v_2 using only red and green vertices.
In this case, starting at v_1, swap red and green.
Then color v red.

Case 2. There is such a path. Similar. ∎