MATH 2602
Linear and Discrete Mathematics
Prof. Margalit
WHAT IS DISCRETE MATH?

Discrete is the opposite of continuous.
WHAT IS DISCRETE MATH?

CONTRERAS
BLADE OF THE FREAK

WHEN YOU FIGHT AGAINST A MACHINE, THE MACHINE DOESN'T HAVE A MIND.

YOU CAN SEE A LIMITED NUMBER OF MOVEMENTS.

Hence, the more complex the machine is, the more movements and combinations it will have.

THUS THE FIGHT WILL DEPEND ON YOUR STRENGTH AND PHYSICAL ABILITY.

IF AN AUTOMATION IS COMPLEX ENOUGH, THE AMOUNT OF STATES AND POSITIONS IS SUCH THAT IT'LL BE IMPOSSIBLE TO SEE THE DIFFERENCE FROM A HUMAN?

I SEE IT'S LIKE THE DIFFERENCE BETWEEN A DIGITAL PICTURE AND AN OPTICAL ONE.

THE PERFECT PICTURE IS FORGOTTEN BY PIXELS IF ONE GETS CLOSER ENOUGH ONE CAN SEE THEM.

IF THE PIXELS ARE TOO SMALL, IT'S IMPOSSIBLE TO SEE THEM AT A SINGLE GLANCE.
What is Discrete Math?

Continuous | Discrete
---|---
real numbers | integers
measuring | counting
ideal shapes | computer images
wave | particle
Differential Eqn | recurrence reln.
calculus | probability
graph theory
algorithms
CHAPTER 5
INDUCTION & RECURSION

Section 5.1
Mathematical Induction
Towers of Hanoi

Proposition: The Towers of Hanoi puzzle with \( n \) disks is solvable.

Proof: First, the puzzle is easily solvable when \( n = 1 \).

Second, if I have a solution to the puzzle with \( n \) disks, then I can easily turn that into a solution for the puzzle with \( n+1 \) disks.

Therefore, I conclude that the puzzle is solvable for \( n=1,2,3,\ldots \) disks!
If I knock the first domino down, and I know each falling domino causes the next one to fall, then I know that all the dominos will fall (even if there are infinitely many).
CLICKER QUESTION

How cute is this kid?

a) very cute
b) extremely cute
c) ridiculously cute
d) cutest kid I have ever seen
The Principle of Mathematical Induction

Say we have a mathematical statement that depends on a natural number \( n \).

Suppose:

1. The statement is true for \( n = n_0 \).
2. Whenever the statement is true for \( n = k \), it is true for \( n = k+1 \).

Then the statement is true for all \( n \geq n_0 \).
HOW TO DO A PROOF BY INDUCTION

First, prove the proposition for a base case $n=n_0$.

Next, assume the proposition is true for $n=k$.

Using the assumption, prove that the proposition is true for $n=k+1$.

By the principle of mathematical induction, conclude that the proposition is true for all $n>n_0$. 
Example

Proposition: For $n \geq 1$:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Proof: The proposition is true for $n=1$:

$$1 = \frac{1(1+1)}{2}$$

Assume the proposition is true for $n = k$, that is:

$$1 + \cdots + k = \frac{k(k+1)}{2}$$

Using the assumption, we will show the proposition is true for $n=k+1$: 

\[ 1 + \cdots + k + 1 = (1 + \cdots + k) + (k+1) \]
\[ = \frac{k(k+1)}{2} + k + 1 \]
\[ = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \]
\[ = \frac{(k+1)(k+2)}{2} \]
\[ = \frac{(k+1)((k+1)+1)}{2} \]

By the principal of mathematical induction, the proposition is proven. \[\square\]
Two Non-Inductive Proofs

\[
\begin{array}{cccccc}
1 & 2 & 3 & \ldots & n-1 & n \\
\frac{n}{n+1} & \frac{n-1}{n+1} & \frac{n-2}{n+1} & \ldots & \frac{2}{n+1} & \frac{1}{n+1}
\end{array}
\]

Total: \( n(n+1) \)  \quad \text{First row:} \quad \frac{n(n+1)}{2}

Area: \( \frac{n(n+1)}{2} \)
**Example**

**Proposition:** \(7^n - 1\) is divisible by 6 for \(n \geq 0\).

**Proof:** First, the proposition is true for \(n = 0\):
\[
7^0 - 1 = 1 - 1 = 0
\]
and 0 is divisible by 6.

Now, assume the proposition is true for \(n = k\):
\[
7^k - 1 \text{ is divisible by 6.}
\]
Using the assumption, we'll show the proposition is true for \(n = k + 1\), that is:
\[
7^{k+1} - 1 \text{ is divisible by 6.}
\]
\[ 7^{k+1} - 1 = 7 \cdot 7^k - 1 \\
= 7 \cdot 7^k - 7 + 6 \\
= 7(7^k - 1) + 6 \]

divisible by 6 by our assumption.

The sum of two numbers divisible by 6 is again divisible by 6.

By the principle of mathematical induction, the proposition is proven. \[ \square \]
More Examples

Proposition: Any debt of \( n \geq 4 \) dollars can be paid with $2 bills and $5 bills.

Proposition: For \( n \geq 1 \):
\[
\frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{1}{2}
\]

Hint: Use +/- trick again.

Proposition: Any \( n \) lines in the plane with no two parallel and no triple intersections divide the plane into \( n(n+1)/2 + 1 \) regions.
Fibonacci Numbers

\[ F_0 = 0 \]
\[ F_1 = 1 \]
\[ F_n = F_{n-1} + F_{n-2} \]

Month
1 2 3 4 5
AN INDUCTION PROBLEM WITH FIBONACCI NUMBERS

PROPOSITION: For $n > 1$:
\[ F_1 + \cdots + F_n = F_{n+2} - 1 \]

Proof: First, the proposition is clearly true for $n = 1$:
\[ F_1 = 1 = F_3 - 1 = 2 - 1 \]

Next, we assume the proposition is true for $n = k$:
\[ F_1 + \cdots + F_k = F_{k+2} - 1 \]

Now, using the assumption, we show the proposition is true for $n = k+1$:
\[ F_1 + \cdots + F_{k+1} = F_{k+1} \]
\[ F_k + \ldots + F_{k+1} = (F_1 + \ldots + F_k) + F_{k+1} \]
\[ = F_{k+2} - 1 + F_{k+1} \]
\[ = (F_{k+1} + F_{k+2}) - 1 \]
\[ = F_{k+3} - 1 \]
\[ = F_{(k+1)+2} - 1 \]

By the principle of mathematical induction, the proposition is proven. 

Leonardo Fibonacci
**More Problems**

**Proposition:** For $n > 0$:

$$F_n = \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

**Proposition:** For $n \geq 1$:

$$F_1 + \ldots + F_{2n-1} = F_{2n}$$

**Proposition:** The number of $n$-digit binary strings with no consecutive 1's is $F_{n+2}$. 