MATH 2602

LINEAR AND DISCRETE MATHEMATICS

PROF. MARGALIT

WHAT IS DISCRETE MATH?

dis·crete [dih-**skreet**] Show IPA

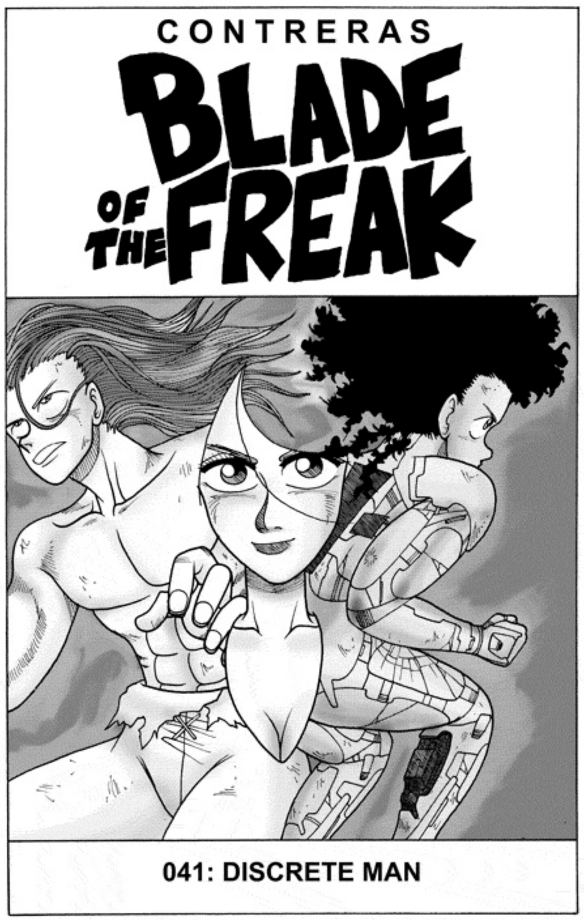
adjective

1. apart or detached from others; separate; distinct: *six discrete parts.*
2. consisting of or characterized by distinct or individual parts; discontinuous.
3. *Mathematics* .
   a. (of a topology or topological space) having the property that every subset is an open set.
   b. defined only for an isolated set of points: *a discrete variable.*
   c. using only arithmetic and algebra; not involving calculus: *discrete methods.*

dictionary.com

Discrete is the opposite of continuous.

WHAT IS DISCRETE MATH?



WHAT IS DISCRETE MATH?

CONTINUOUS	DISCRETE
real numbers	integers
measuring	counting
ideal shapes	computer images
wave	particle
differential eqn	recurrence reln.
calculus	probability
	graph theory
	algorithms

CHAPTER 5
INDUCTION & RECURSION

Section 5.1
Mathematical Induction

TOWERS OF HANOI

Proposition: The Towers of Hanoi puzzle with n disks is solvable.

Proof: First, the puzzle is easily solvable when $n = 1$.

Second, if I have a solution to the puzzle with n disks, then I can easily turn that into a solution for the puzzle with $n+1$ disks.

Therefore, I conclude that the puzzle is solvable for $n = 1, 2, 3, \ldots$ disks!

Dominos

If I knock the first domino down, and I know each falling domino causes the next one to fall, then I know that <u>all</u> the dominos will fall (even if there are infinitely many).

CLICKER QUESTION

How cute is this kid?

a) very cute

b) extremely cute

c) ridiculously cute

d) cutest kid I have ever seen

THE PRINCIPLE OF MATHEMATICAL INDUCTION

Say we have a mathematical statement that depends on a natural number n.
Suppose:

1. The statement is true for $n = n_0$.
2. Whenever the statement is true for $n = k$, it is true for $n = k+1$.

Then the statement is true for all $n \geq n_0$.

HOW TO DO A PROOF BY INDUCTION

First, prove the proposition for a <u>base</u> <u>case</u> $n = n_0$

Next, assume the proposition is true for $n = k$

<u>Using</u> the assumption, prove that the proposition is true for $n = k+1$.

By the principle of mathematical induction, Conclude that the proposition is true for all $n \geq n_0$.

EXAMPLE

PROPOSITION: For $n \geq 1$:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

PROOF: The proposition is true for $n = 1$:

$$1 = \frac{1(1+1)}{2}.$$

Assume the proposition is true for $n = k$, that is:

$$1 + \cdots + k = \frac{k(k+1)}{2}$$

Using the assumption, we will show the proposition is true for $n = k+1$:

$$1+\cdots+k+1=(1+\cdots+k)+(k+1)$$

$$=\frac{k(k+1)}{2}+k+1$$

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$$

$$=\frac{(k+1)(k+2)}{2}$$

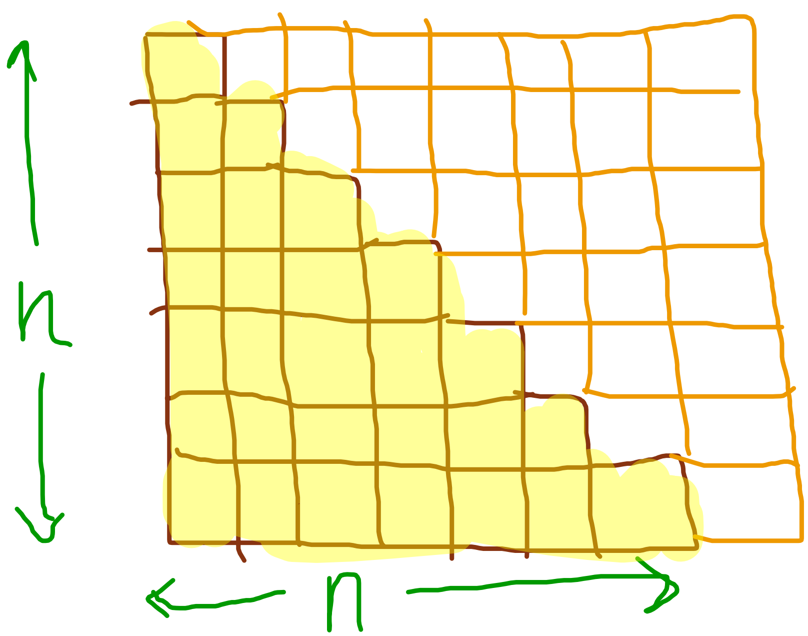
$$=\frac{(k+1)((k+1)+1)}{2}$$

By the principal of mathematical induction, the proposition is proven. $\blacksquare$

Two Non-Inductive Proofs

$$\begin{array}{ccccccc}
 & 1 & 2 & 3 & \cdots & n-1 & n \\
+ & n & n-1 & n-2 & \cdots & 2 & 1 \\
\hline
 & n+1 & n+1 & n+1 & & n+1 & n+1
\end{array}$$

Total: $n(n+1)$ First row: $\frac{n(n+1)}{2}$



Area: $\frac{n(n+1)}{2}$

EXAMPLE

PROPOSITION: $7^n - 1$ is divisible by 6 for $n \geq 0$.

PROOF: First, the proposition is true for $n=0$:

$$7^0 - 1 = 1 - 1 = 0$$

and 0 is divisible by 6.

Now, assume the proposition is true for $n=k$:

$7^k - 1$ is divisible by 6.

Using the assumption, we'll show the proposition is true for $n=k+1$, that is: $7^{k+1} - 1$ is divisible by 6.

$$
\begin{aligned}
7^{k+1} - 1 &= 7 \cdot 7^k - 1 \\
&= 7 \cdot 7^k - 7 + 6 \\
&= 7(7^k - 1) + 6
\end{aligned}
$$

divisible by 6 by our assumption.

The sum of two numbers divisible by 6 is again divisible by 6.

By the principle of mathematical induction, the the proposition is proven. $\blacksquare$

MORE EXAMPLES

PROPOSITION: Any debt of $n \geq 4$ dollars can be paid with \$2 bills and \$5 bills.

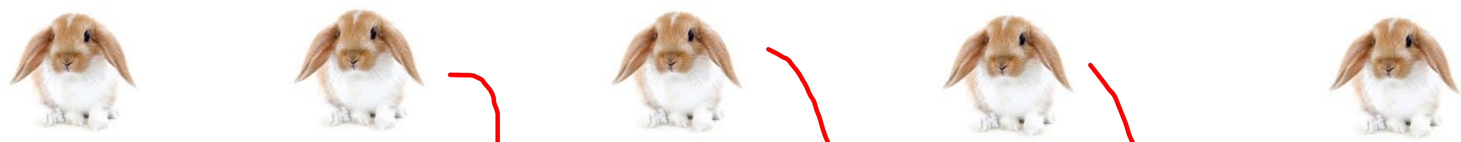
PROPOSITION: For $n \geq 1$:

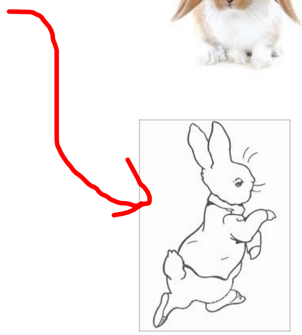
$$\frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{1}{2}$$

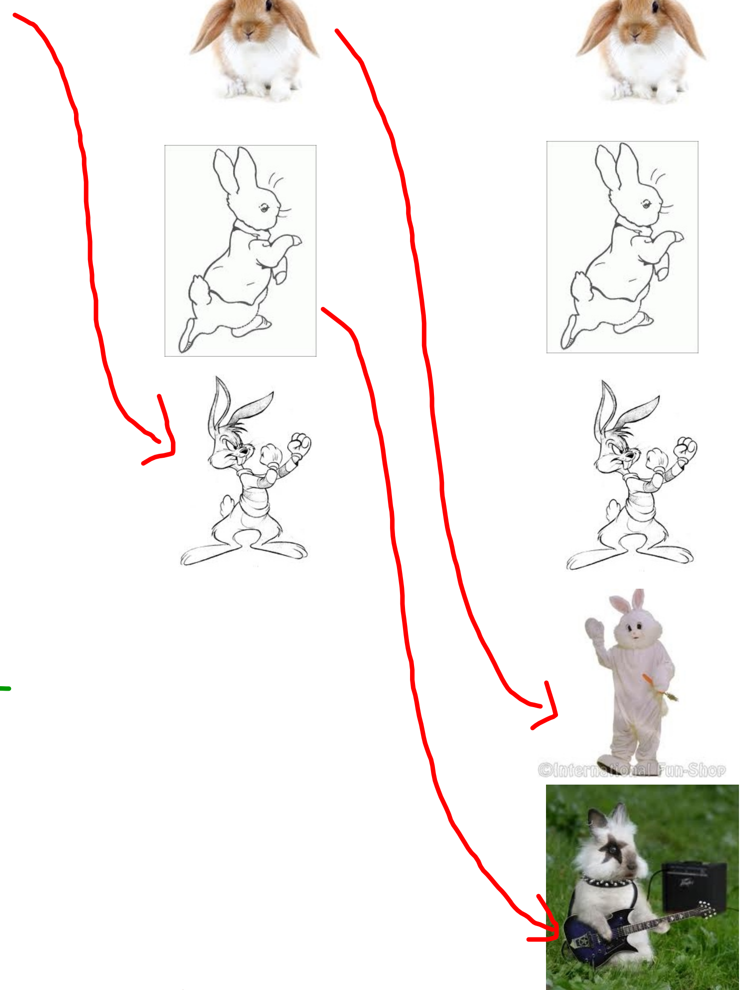
Hint: Use +/- trick again.

PROPOSITION: Any n lines in the plane with no two parallel and no triple intersections divide the plane into $n(n+1)/2 + 1$ regions.

Fibonacci Numbers







$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}$$

Month

1 2 3 4 5

An Induction Problem with Fibonacci Numbers

Proposition: For $n \geq 1$:

$$F_1 + \cdots + F_n = F_{n+2} - 1$$

Proof: First, the proposition is clearly true for $n=1$:

$$F_1 = 1 = F_3 - 1 = 2 - 1$$

Next, we assume the proposition is true for $n=k$:

$$F_1 + \cdots + F_k = F_{k+2} - 1$$

Now, using the assumption, we show the proposition is true for $n=k+1$:

$$F_1 + \cdots + F_{k+1} = F_{k+1}$$

$$
\begin{aligned}
F_1 + \cdots + F_{k+1} &= (F_1 + \cdots F_k) + F_{k+1} \\
&= F_{k+2} - 1 + F_{k+1} \\
&= (F_{k+1} + F_{k+2}) - 1 \\
&= F_{k+3} - 1 \\
&= F_{(k+1)+2} - 1
\end{aligned}
$$

By the principle of mathematical induction, the proposition is proven. $\blacksquare$

Leonardo Fibonacci

More Problems

Proposition: For $n \geq 0$:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Proposition: For $n \geq 1$:

$$F_1 + \cdots + F_{2n-1} = F_{2n}$$

Proposition: The number of n-digit binary strings with no consecutive 1's is F_{n+2}.