SECTION 7.3
Elementary Probability

INTUITIVE PROBABILITY

What is the probability that...

a) A flipped coin comes up heads?

$1/2$

b) A rolled die comes up 3?

$1/6$

c) A rolled pair of dice comes up 4?

$3/36 = 1/12$

DEFINITIONS

An experiment is a procedure that yields one of a given set of outcomes.

The sample space of the experiment is the set of possible outcomes.

S = finite set

Blaise Pascal

Pierre Laplace

An event is a subset of the sample space:

$$A \subseteq S$$

The probability of an event A, assuming each outcome of the experiment is equally likely, is:

$$P(A) = |A|/|S|$$

EXAMPLES OF EXPERIMENTS

Experiment	Sample space S	Outcome A	Probability $P(A)$
Flipping a coin	$\{H, T\}$	$\{H\}$	$1/2$
Rolling a die	$\{1,2,3,4,5,6\}$	$\{3\}$	$1/6$
Rolling a pair of dice	$\{(1,1),(1,2),\ldots(6,5),(6,6)\}$	$\{(1,3),(2,2),(3,1)\}$	$3/36 = 1/12$

MORE EXAMPLES

1. You toss a coin 5 times. What is the probability of getting 4 heads?

$|S| = 2^5 = 32$
$A = \{HHHHT, HHHTH, \ldots, THHHH\}$
$P(A) = 5/32.$

2. What is the probability of correctly guessing the winners in a 64-team single elimination tournament?
(Assume every team has a 50% chance of winning each game.

$|S| = 2^{63} = 18,446,744,073,709,551,616$
$P(A) = 1/2^{63}$

Population of Earth: 7 billion.

MORE EXAMPLES

3. An urn has 4 red balls, 3 green balls. You pull one ball at random. What is the probability of pulling a green ball?

$S \neq \{R, G\}$

$S = \{R1, R2, R3, R4, G1, G2, G3\}$

$A = \{G1, G2, G3\}$

$\rightsquigarrow P(A) = 3/7$

Suppose you pull one ball, replace it, then pull another ball. What is the probability of pulling two balls of the same color?

$|S| = 49$

$|A| = 9 + 16 = 25$

$\rightsquigarrow P(A) = 25/49$

MORE EXAMPLES

Same urn (4 red, 3 green). Now suppose you pull one ball, don't replace it, and pull another ball. What is the probability of getting two balls of the same color?

First way: pull one at a time

$$|S| = 7 \cdot 6 = 42$$

$$|A| = 3 \cdot 2 + 4 \cdot 3 = 18$$

$$\rightsquigarrow P(A) = 18/42 = 3/7$$

Second way: pull two at once

$$|S| = \binom{7}{2} = 7 \cdot 6/2 = 21$$

$$|A| = \binom{4}{2} + \binom{3}{2} = 6 + 3 = 9$$

$$\rightsquigarrow P(A) = 9/21 = 3/7$$

MORE EXAMPLES

4. In poker, what is the probability of dealing a 4-of-a-kind?

$S = \{\text{poker hands}\} = \binom{52}{5} = 2{,}598{,}960$

$A = \{\text{4-of-a-kind hands}\}$

What is $|A|$?

Pick a kind: 4

Pick 4 of that kind: 1

Pick a 5^{th} card: 48

$\rightsquigarrow P(A) = {}^{13 \cdot 48}/_{2{,}598{,}960} \approx .00024 \approx {}^{1}/_{4000}$

What about a full house?

S = same

What is $|A|$?

Pick 1^{st} kind (13), 2^{nd} kind (12),

3 of 1^{st} kind $\binom{4}{3}$, 2 of 2^{nd} $\binom{4}{2}$

$\rightsquigarrow P(A) = {}^{3744}/_{2{,}598{,}960} \approx .0014 \approx {}^{1}/_{700}$

SOME PROBABILITY RULES

THEOREM: Let S be the sample space of some experiment.
Let A and B be events.

(i) $0 \leq P(A) \leq 1$

$P(\emptyset) = 0, \quad P(S) = 1$

(ii) $P(A^c) = 1 - P(A)$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally:

$$P(A_1 \cup \cdots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots$$

Can rephrase all counting rules as probability rules.

APPLYING PROBABILITY RULES

EXAMPLE: A number from 1 to 100 is chosen at random. What is the probability it is...

a) divisible by 2, 3, or 5?
b) divisible by 2 and 3, but not 5?
c) divisible by 3 but not 2 or 5?
d) divisible by at most two of 2, 3, and 5?

$|S| = 100$

$A_k = \{1 \leq n \leq 100 : n \text{ is divisible by } k\}$

$P(A_k) = \lfloor \frac{100}{k} \rfloor / 100$

$A_j \cap A_k = A_{\mathrm{lcm}(j,k)}$ so if $\gcd(j,k) = 1$ then $A_j \cap A_k = A_{jk}$

a) $P(A_2 \cup A_3 \cup A_5) = 74/100$

b) $P((A_2 \cap A_3) \setminus A_5) = 13/100$

c) $P(A_3 \setminus (A_2 \cup A_5)) = 12/100$

d) $P((A_2 \cap A_3 \cap A_5)^c) = 97/100$

Mutual Exclusivity

Two events A and B are mutually exclusive if $A \cap B = \emptyset$

Events $A_1, \ldots, A_n$ are pairwise mutually exclusive if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

A special case of the last theorem:

If $A_1, \ldots, A_n$ are pairwise mutually exclusive events, then

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n) \quad \text{(addition rule)}$$

Example: A number from 1 to 100 is chosen at random. What is the probability that the number is divisible by 7 or 30?

$$A_7 \cap A_{30} = A_{210} = \emptyset \rightsquigarrow P(A_7 \cup A_{30}) = P(A_7) \cup P(A_{30}) =$$

$$14/100 + 3/100 = 17/100$$

APPLYING PROBABILITY RULES

1. What is the probability that a length 10 bit string (chosen at random) has at least one zero? at least two zeros?

2. What is the probability that a poker hand (dealt at random) is a flush? a straight? royal flush?

Note: A, 2, 3, 4, 5 and 10, J, Q, K, A are both straights.