SECTION 7.3
Elementary Probability
INTUITIVE PROBABILITY

What is the probability that...

a) A flipped coin comes up heads?

\[ \frac{1}{2} \]

b) A rolled die comes up 3?

\[ \frac{1}{6} \]

c) A rolled pair of dice comes up 4?

\[ \frac{3}{36} = \frac{1}{12} \]
DEFINITIONS

An experiment is a procedure that yields one of a given set of outcomes.

The sample space of the experiment is the set of possible outcomes.

\[ S = \text{finite set} \]

An event is a subset of the sample space:

\[ A \subseteq S \]

The probability of an event \( A \), assuming each outcome of the experiment is equally likely, is:

\[ P(A) = \frac{|A|}{|S|} \]
### Examples of Experiments

<table>
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<tr>
<th>Experiment</th>
<th>Sample space $S$</th>
<th>Outcome $A$</th>
<th>Probability $P(A)$</th>
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<td>Flipping a coin</td>
<td>${H, T}$</td>
<td>${H}$</td>
<td>$\frac{1}{2}$</td>
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<tr>
<td>Rolling a die</td>
<td>${1, 2, 3, 4, 5, 6}$</td>
<td>${3}$</td>
<td>$\frac{1}{6}$</td>
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<td>Rolling a pair of dice</td>
<td>${(1,1), (1,2), \ldots, (6,5), (6,6)}$</td>
<td>${(1,3), (2,2), (3,1)}$</td>
<td>$\frac{3}{36} = \frac{1}{12}$</td>
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</table>
MORE EXAMPLES

1. You toss a coin 5 times. What is the probability of getting 4 heads?

\[ |S| = 2^5 = 32 \]
\[ A = \{ \text{HHHHT, HHHTH, \ldots, TTHHHH} \} \]
\[ P(A) = \frac{5}{32}. \]

2. What is the probability of correctly guessing the winners in a 64-team single elimination tournament?

(Assume every team has a 50% chance of winning each game)

\[ |S| = 2^{63} = 18,446,744,073,709,551,616 \]
\[ P(A) = \frac{1}{2^{63}} \]

Population of Earth: 7 billion.
MORE EXAMPLES

3. An urn has 4 red balls, 3 green balls. You pull one ball at random. What is the probability of pulling a green ball?

\[
S = \{R,G\}
\]
\[
S = \{R_1, R_2, R_3, R_4, G_1, G_2, G_3\}
\]
\[
A = \{G_1, G_2, G_3\}
\]
\[
P(A) = \frac{3}{7}
\]

Suppose you pull one ball, replace it, then pull another ball. What is the probability of pulling two balls of the same color?

\[
|S| = 49
\]
\[
|A| = 9 + 16 = 25
\]
\[
P(A) = \frac{25}{49}
\]
MORE EXAMPLES

Same urn (4 red, 3 green). Now suppose you pull one ball, don’t replace it, and pull another ball. What is the probability of getting two balls of the same color?

First way: pull one at a time
\[ |S| = 7 \cdot 6 = 42 \]
\[ |A| = 3 \cdot 2 + 4 \cdot 3 = 18 \]
\[ \sim P(A) = \frac{18}{42} = \frac{3}{7} \]

Second way: pull two at once
\[ |S| = \binom{7}{2} = \frac{7 \cdot 6}{2} = 21 \]
\[ |A| = \binom{4}{2} + \binom{3}{2} = 6 + 3 = 9 \]
\[ \sim P(A) = \frac{9}{21} = \frac{3}{7} \]
MORE EXAMPLES

4. In poker, what is the probability of dealing a 4-of-a-kind?

\[ S = \{ \text{poker hands} \} = \binom{52}{5} = 2,598,960 \]

\[ A = \{ \text{4-of-a-kind hands} \} \]

What is \(|A|\)?

Pick a kind: 4

Pick 4 of that kind: 1

Pick a 5th card: 48

\[ P(A) = \frac{13 \cdot 48}{2,598,960} \approx 0.00024 \approx \frac{1}{4000} \]

What about a full house?

\[ S = \text{same} \]

What is \(|A|\)?

Pick 1st kind (13), 2nd kind (12),

3 of 1st kind (4), 2 of 2nd (4)

\[ P(A) = \frac{3744}{2,598,960} \approx 0.0014 \approx \frac{1}{700} \]
SOME PROBABILITY RULES

**Theorem:** Let $S$ be the sample space of some experiment. Let $A$ and $B$ be events.

(i) $0 \leq P(A) \leq 1$
    
    $P(\emptyset) = 0$, $P(S) = 1$

(ii) $P(A^c) = 1 - P(A)$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally:

$$P(A_1 \cup \cdots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots$$

Can rephrase all counting rules as probability rules.
APPLYING PROBABILITY RULES

Example: A number from 1 to 100 is chosen at random.
What is the probability it is...
   a) divisible by 2, 3, or 5?
   b) divisible by 2 and 3, but not 5?
   c) divisible by 3 but not 2 or 5?
   d) divisible by at most two of 2, 3, and 5?

|S| = 100
A_k = \{1 \leq n \leq 100 : n \text{ is divisible by } k\}
P(A_k) = \left[\frac{100}{k}\right]/100
A_j \cap A_k = A_{lcm(j,k)}\text{ so if } \gcd(j,k) = 1 \text{ then } A_j \cap A_k = A_{jk}

a) P(A_2 \cup A_3 \cup A_5) = \frac{74}{100}  \quad c) P(A_3 \setminus (A_2 \cup A_5)) = \frac{12}{100}
b) P((A_2 \cap A_3) \setminus A_5) = \frac{13}{100}  \quad d) P((A_2 \cap A_3 \cap A_5)^c) = \frac{97}{100}
**Mutual Exclusivity**

Two events $A$ and $B$ are mutually exclusive if $A \cap B = \emptyset$.

Events $A_1, \ldots, A_n$ are pairwise mutually exclusive if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

A special case of the last theorem:

If $A_1, \ldots, A_n$ are pairwise mutually exclusive events, then

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n) \quad \text{(addition rule)}$$

**Example:** A number from 1 to 100 is chosen at random. What is the probability that the number is divisible by 7 or 30?

$$A_7 \cap A_{30} = A_{210} = \emptyset \implies P(A_7 \cup A_{30}) = P(A_7) + P(A_{30}) = \frac{14}{100} + \frac{3}{100} = \frac{17}{100}$$
APPLYING PROBABILITY RULES

1. What is the probability that a length 10 bit string (chosen at random) has at least one zero? at least two zeros?

2. What is the probability that a poker hand (dealt at random) is a flush? a straight? royal flush?

Note: A, 2, 3, 4, 5 and 10, J, Q, K, A are both straights.