SECTION 7.4
Probability Theory

DEFINITIONS

An experiment is a procedure that yields one of a given set of outcomes.

The sample space of the experiment is the set of possible outcomes.

S = finite set


Blaise Pascal


Pierre Laplace

An event is a subset of the sample space:

$$A \subseteq S$$

The probability of an event A, assuming each outcome of the experiment is equally likely, is:

$$P(A) = |A|/|S|$$

PROBABILITY FUNCTIONS

Say we do an experiment with outcomes $s_1, \dots, s_n$. It might be that the s_i are not equally likely. For instance, consider an unfair die:

$$P(1) = 1/3$$
$$P(2) = P(3) = 1/12$$
$$P(4) = P(5) = P(6) = 1/6$$

What is the probability of rolling an even number?

$$P(2) + P(4) + P(6) = 1/12 + 1/6 + 1/6 = 5/12$$

Odd?

$$1 - 5/12 = 7/12$$

A 4, 5, or 6?

$$P(4) + P(5) + P(6) = 1/6 + 1/6 + 1/6 = 1/2$$

PROBABILITY FUNCTIONS

For an experiment with outcomes $S = \{s_1, \dots, s_n\}$, a probability function is a function

$$P : S \to \mathbb{R}$$

with (i) $0 \le P(s_i) \le 1$ for all i. (the ≤ 1 is redundant)

(ii) $P(s_1) + \cdots + P(s_n) = 1$

If $A \subseteq S$ is an event, then

$$P(A) = \sum_{s_i \in A} P(s_i)$$

If each s_i is equally likely, then $P(s_i) = 1/|S|$

so $P(A) = \sum_{s_i \in A} \frac{1}{|S|} = |A|/|S|$, as before

Still true that:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) $P(A^c) = 1 - P(A)$

THE MONTY HALL PROBLEM

"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to take the switch?"

THE MONTY HALL PROBLEM

Intuitive explanation: There are 3 equally likely cases:

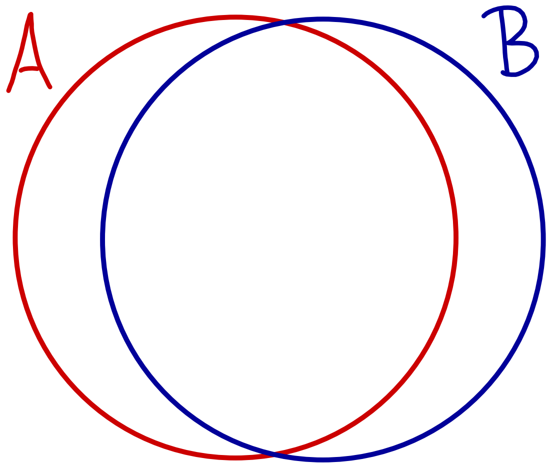
1. You initially picked a car ⇝ switching gives a goat
2. You initially picked goat #1 ⇝ switching gives a car
3. You initially picked goat #2 ⇝ switching gives a car

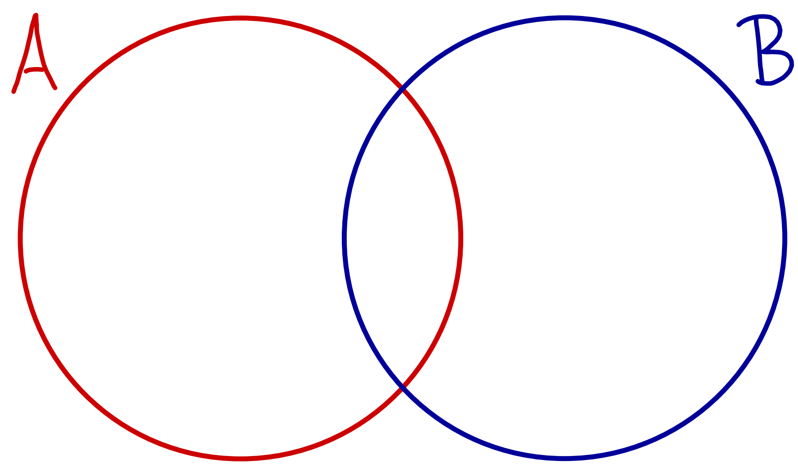
So in $2/3$ of the cases, switching gives a car.

CONDITIONAL PROBABILITY

Say A and B are events and $P(A) > 0$. The conditional probability of B given A is:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$





$P(B|A)$ large

$P(B|A)$ small

$P(B)$ is same in both, but the knowledge of being in A makes a big difference.

CONDITIONAL PROBABILITY EXAMPLE

A coin is flipped twice. The first flip is heads. What is the probability that both flips are heads?

Intuition: $1/2$

Basic probability: $S = \{HH, HT\}$
$A = \{HH\}$
$\rightsquigarrow P(A) = 1/2$

Conditional probability: $P(HH \mid H*) = \dfrac{P(HH \cap H*)}{P(H*)} = \dfrac{1/4}{2/4}$

HH = both flips heads
H* = first flip heads.

CONDITIONAL PROBABILITY EXAMPLE

I have two kids. One is a boy. What is the probability I have two boys?

S:

BB	BG
GB	GG

B = 2 boys

A = at least 1 boy

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{3/4} = 1/3$$

CONDITIONAL PROBABILITY EXAMPLES

1. An urn has 10 white, 5 yellow, and 10 black marbles. A marble is chosen at random. We are told it is not black. What is the probability it is yellow?

$$P(Y|B^c) = \frac{P(Y \cap B^c)}{P(B^c)} = \frac{5/25}{15/25} = 1/3$$

2. We deal bridge hands at random to N, S, E, W. Together, N and S have 8 spades. What is the probability that E has 3 spades?

We know E & W have 5 spades.

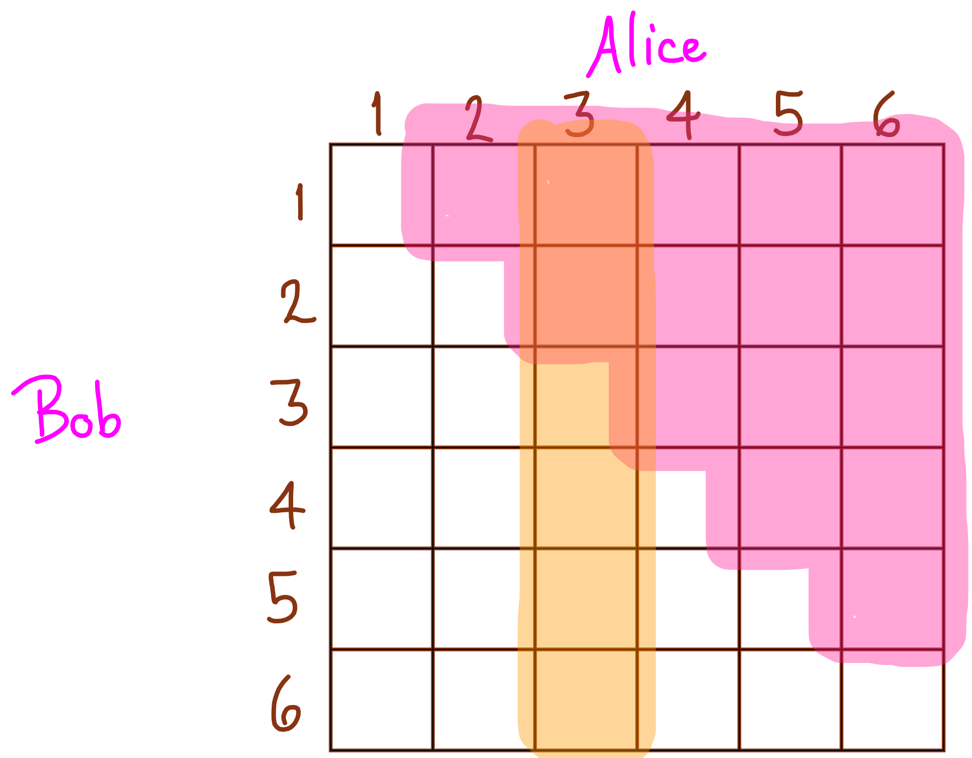
B = E has 3 spades, A = E & W have 5 spades

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\binom{5}{3}\binom{21}{10}/N}{\binom{26}{13}/N} = \frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}}$$

N = total # of ways of dealing hands.

CONDITIONAL PROBABILITY EXAMPLE

Alice and Bob each roll a die. We are told that Alice rolled a higher number. What is the probability that Alice rolled a 3?



B = Alice rolled 3

A = Alice > Bob

$$P(B|A) = P(B \cap A) / P(A) = \frac{2/36}{15/36} = 2/15.$$

INDEPENDENCE

Events A and B are *independent* if

$$P(B|A) = P(B)$$

Since $P(B) = \frac{P(B \cap A)}{P(A)}$ we can say A and B are independent if:

$$P(A \cap B) = P(A)P(B)$$

Examples. 1. We roll two die. A = first comes up 2
B = second comes up 3

$$P(B|A) = \frac{1/36}{1/6} = 1/6 = P(B) \rightsquigarrow \text{independent.}$$

$$P(A \cap B) = 1/36 = 1/6 \cdot 1/6 = P(A)P(B) \rightsquigarrow \text{independent.}$$

2. Two kids. B = 2 boys
A = at least one boy

$$P(B|A) = 1/3 \neq 1/4 = P(B) \rightsquigarrow \text{not independent.}$$

INDEPENDENCE

Events A and B are independent if

$$P(B|A) = P(B)$$

Examples. 3. The Alice and Bob problem:

B = Alice rolled 3

A = Alice > Bob

$P(B|A) = 2/15$, $P(B) = 1/6$ $\rightsquigarrow$ not independent

4. Urn problem: 10 white, 5 yellow, 10 black.

Are Y and B^c independent?

$$P(Y|B^c) = 1/3 \neq 1/5 = P(Y)$$

$\rightsquigarrow$ not independent.

A CONDITIONAL PROBABILITY PROBLEM

We buy light bulbs from suppliers A and B.

30% of the bulbs come from A, 70% from B.

2% of the bulbs from A are defective

3% of the bulbs from B are defective.

What is the probability that a random bulb...

(i) is from A and defective?

(ii) is from B and not defective?

☆ (iii) is defective?

Given: $P(A) = 3/10$

$P(B) = 7/10$

$P(D|A) = 2/100$

$P(D|B) = 3/100$

(i) $P(A \cap D) = P(A)P(D|A) = \frac{3}{10} \cdot \frac{2}{100} = 3/500$

(ii) $P(B \cap D^c) = P(B) - P(B \cap D)$

$= P(B) - P(B)P(D|B)$

$= 7/10 - 7/10 \cdot 3/100 = 679/1000$

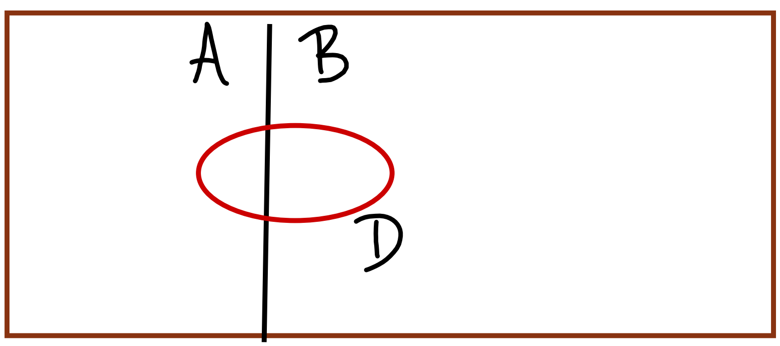
(iii) $P(D) = P(D \cap A) + P(D \cap B)$

$= P(A)P(D|A) + P(B)P(D|B) = 27/1000$

A CONDITIONAL PROBABILITY PROBLEM

We buy light bulbs from suppliers A and B.
30% of the bulbs come from A, 70% from B.
2% of the bulbs from A are defective
3% of the bulbs from B are defective.

What is the probability that a random bulb...
(i) is from A and defective?
(ii) is from B and not defective?
☆ (iii) is defective?



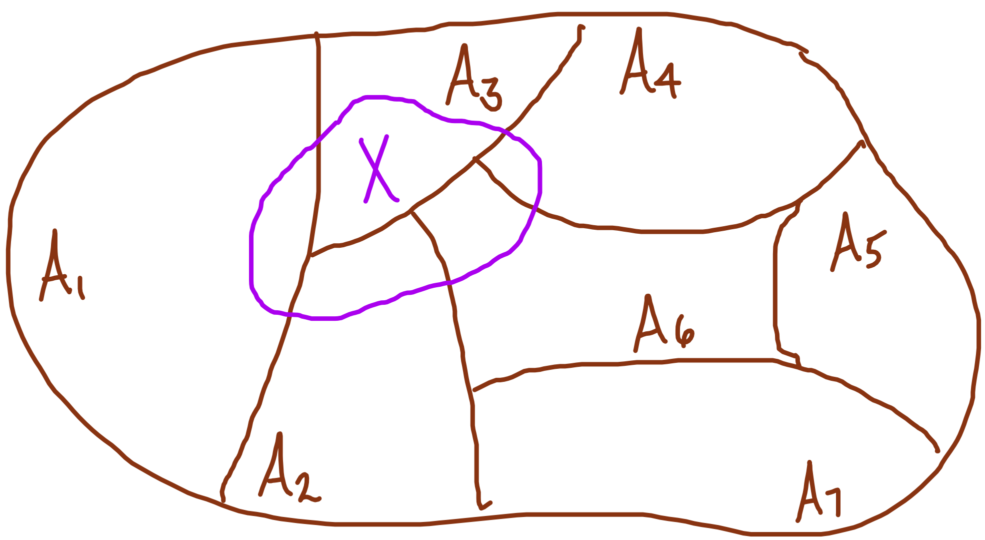
Reinterpret all questions in terms of areas.

LAW OF TOTAL PROBABILITY

Say that events $A_1, \ldots, A_n$ form a partition of the sample space S, that is, the A_i are mutually exclusive ($A_i \cap A_j = \emptyset$ for $i \neq j$) and $A_1 \cup \cdots \cup A_n = S$.

Let $X \subseteq S$ be any event. Then

$$P(X) = P(A_1)P(X|A_1) + \cdots + P(A_n)P(X|A_n)$$



BAYES' FORMULA

How is $P(A|B)$ related to $P(B|A)$?

THEOREM: $$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$

PROOF: $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{P(A)} \quad \blacksquare$$

EXAMPLE. In the light bulb problem, say a randomly selected light bulb is defective. What is the probability it came from A?

$$P(A|D) = \frac{P(A)P(D|A)}{P(D)} = \frac{3/10 \cdot 2/100}{27/1000} = 2/9$$

BAYES' FORMULA

EXAMPLE. Coin A comes up heads 1/4 of the time.
Coin B comes up heads 3/4 of the time.
We choose a coin at random and flip it twice.
If we get two heads, what is the probability coin B was chosen?

$$P(B|HH) = \frac{P(B)P(HH|B)}{P(HH)} = \frac{1/2 \cdot 9/16}{P(HH)}$$

To find P(HH), we use the law of total probability:

$$\begin{aligned} P(HH) &= P(HH|A)P(A) + P(HH|B)P(B) \\ &= 1/16 \cdot 1/2 + 9/16 \cdot 1/2 \\ &= 10/32 \end{aligned}$$

$$\rightsquigarrow P(B|HH) = \frac{9/32}{10/32} = 9/10$$

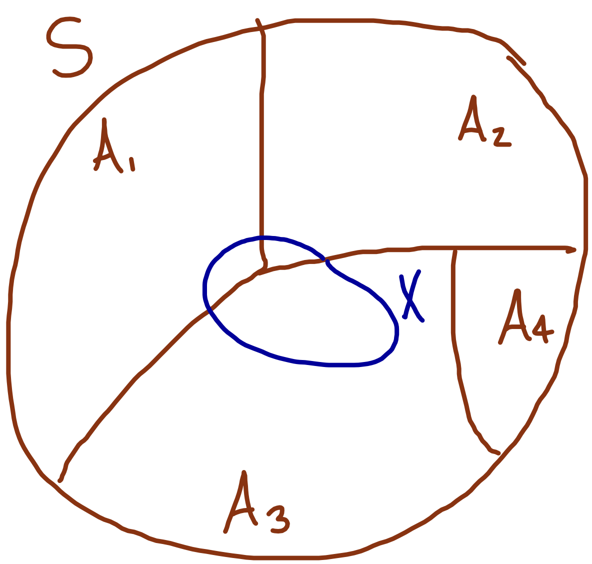
BAYES' FORMULA

Computing the denominator with the law of total probability

$A_1, \dots, A_n$ pairwise mutually exclusive events with $A_1 \cup \cdots \cup A_n = S$ and $P(A_i) > 0$ for all i. Let X be an event with $P(X) > 0$. Then, for each j, we have:

$$P(A_j | X) = \frac{P(A_j) P(X | A_j)}{P(X)}$$

where $P(X) = P(A_1) P(X | A_1) + \cdots + P(A_n) P(X | A_n)$



$P(A_3 | X)$ big

$P(A_2 | X)$ small

$P(A_4 | X) = 0.$

EXAMPLE. Do a variant of the coin problem with 3 or more coins.

BAYES' FORMULA

PROBLEM. You have 3 cards. One is red on both sides, one is black on both sides, and one has a red side and a black side. You pick one card randomly and put it on the table. Its top side is red. What is the probability the other side is red?

$$P(RR|R) = \frac{P(R|RR)P(RR)}{P(R|RR)P(RR) + P(R|RB)P(RB) + P(R|BB)P(BB)}$$

$$= \frac{1 \cdot 1/3}{1 \cdot 1/3 + 1/2 \cdot 1/3 + 0 \cdot 1/3}$$

$$= \frac{1/3}{1/2} = 2/3$$

This problem is logically equivalent to the Monty Hall problem:
A, B, C = prize behind door A B C, You pick door A.
Door C opened, revealing goat. Want $P(B|\text{not } C)$.

BAYES' FORMULA

PROBLEM. There are 3 urns, A, B, and C that have 2, 4, and 8 red marbles and 8, 6, and 2 black marbles, respectively. A random card is picked from a deck. If the card is black we choose a marble from A, if it is a diamond we choose a marble from B, and otherwise choose a marble from C.

(a) What is the probability that a red marble gets drawn?

$2/5$

(b) If we know a red marble was drawn, what is the probability the card was hearts? diamonds?

$1/2$, $1/4$

Draw the picture!