Section 7.5
Repetitions

REPETITIONS

QUESTION: How many ways are there to put r identical marbles into n boxes, if you are allowed to put more than one marble per box?

First try 3 marbles into 10 boxes.

Case 1: All in same box $\binom{10}{1}$

Case 2: Two in one box, one in another $10 \cdot 9$

Case 3: All different boxes $\binom{10}{3} = 120$

Addition rule $\rightsquigarrow 120 + 90 + 10 = 220.$

What about 10 marbles in 3 boxes?

Lots of cases!

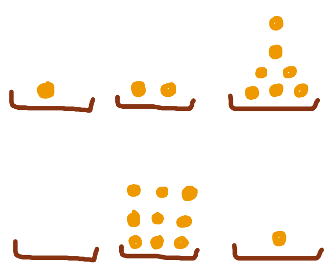
What to do?

Stars and Bars

Can answer the last question by looking at it the right way:

The number of ways of putting 10 marbles into 3 boxes is the same as:

the number of binary strings with 10 zeros, 2 ones
(or 10 stars, 2 bars)



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How many such strings are there?

$\binom{12}{2} = 66$ (choose which of the 12 spots will be stars.)

REPETITIONS

QUESTION: How many ways are there to put r identical marbles into n boxes, if you are allowed to put more than one marble per box?

ANSWER: This is the same as the number of strings with r stars and $n-1$ bars:

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Repetitions, Permutations, and Combinations

How many ways to put r marbles in n boxes if...

	the marbles are indistinguishable	the marbles are distinguishable
at most one marble is allowed per box	$\binom{n}{r}$	$P(n,r)$
any number of marbles is allowed in a box	$\binom{n+r-1}{r}$	n^r

REPETITIONS

EXAMPLE: How many ways are there to choose 15 cans of soda from a cooler with (lots of) Coke, Dr. Pepper, Mtn Dew, RC cola, and Mr. Pibb?

Idea: Put 15 marbles in 5 boxes

$$\rightsquigarrow \binom{19}{15} = \binom{19}{4} = 3876$$

FURTHER: What if I insist on at least 3 Cokes and exactly one Mr. Pibb?

4 "marbles" already "used." One "box" already "full." $\rightsquigarrow \binom{14}{3} = 14 \cdot 13 \cdot 12 / 6 = 364$

Repetitions

Example. In how many ways can we choose 4 nonnegative integers a, b, c, and d so that $a+b+c+d=100$?

100 marbles in 4 boxes: $\binom{103}{3} = 176,851$

What if a, b, c, and d are natural numbers?

4 marbles used $\rightsquigarrow \binom{99}{3} = 156,849$

REPETITIONS

EXAMPLE. How many ways are there to choose 4 integers $a, b, c,$ and d so that:

$$a+b+c+d=15$$

$$a \geq -3,\ b \geq 0,\ c \geq -2,\ d \geq -1\ ?$$

Set $a' = a+3,\ c' = c+2,\ d' = d+1$

$\rightsquigarrow a'+b+c'+d = 21$ where $a', b, c', d \geq 0.$

$\rightsquigarrow$ 21 marbles, 4 boxes: $\binom{24}{3} = 2024$

Generalized Permutations

Example. How many ways are there to arrange the letters of SYZYGY?

Method 1: Choose 3 spots for Y: $\binom{6}{3}$
Then order the other letters: $3!$
120

Method 2: Arrange $SY_1ZY_2GY_3$ in $6!$ ways
Divide by $3!$
120.

Example. What about MISSISSIPPI?

34,650

Generalized Permutations

In general, say we have n objects that fall into k groups, with n_i objects in the i^{th} group. Two objects in the same group are indistinguishable, but objects in different groups are distinguishable. In how many ways can we order the objects?

$$P(n; n_1, \dots, n_k) = n! / n_1! \, n_2! \cdots n_k!$$

This is also the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in $(x_1 + x_2 + \cdots + x_k)^n$

Generalized Permutations

Example. Suppose there are 100 spots in the showroom of a car dealership. There are 15 (identical) sports cars, 25 compact cars, 30 station wagons, and 20 vans. In how many ways can the cars be parked?

$$P(100;15,25,30,20,\underbrace{10}_{\text{blanks}}) = {}^{100!}/_{15!\,25!\,30!\,20!\,10!}$$