SECTION 7.6
Derangements

A Curious Probability

Question. A professor hands back exams randomly. What is the probability that no student gets their own exam?

Answer.

5 students ~ 36.8 %
10 students ~ 36.8 %
100 students ~ 36.8 %

DERANGEMENTS

A derangement of n objects that have some natural order is a rearrangement of the objects so that no object is in its correct position.

QUESTION. How many are there? Call the number D_n.

n	D_n	$P(D_n)$
1	0	0
2	1	$1/2$
3	2	$1/3$
4	9	$3/8$

What is the pattern?

A Formula for D_n

Let A_k be the permutations of n ordered objects with object k in the correct spot.

$$D_n = \left(\bigcup_{k=1}^{n} A_k\right)^c$$

$$\begin{aligned}
D_4 &= 24 - |A_1 \cup A_2 \cup A_3 \cup A_4| \\
&= 24 - \sum |A_i| + \sum |A_i \cap A_j| - |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4| \\
&= 24 - \binom{4}{1}3! + \binom{4}{2}2! + \binom{4}{3}1! + \binom{4}{4}0! \\
&= 9
\end{aligned}$$

$$\begin{aligned}
D_4 &= 4! - 4 \cdot 3! + \frac{4!}{2!2!}2! - \frac{4!}{3!} + 1) \\
&= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)
\end{aligned}$$

Theorem. $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

D_N AND e

THEOREM. $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

Recall: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$\rightsquigarrow e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718$

$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$

$\approx 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$

So $D_n \approx n!/e$

$\rightsquigarrow P(D_n) \approx \frac{n!/e}{n!} = 1/e \approx 0.368$

For $n \geq 5$, this is correct to 3 decimal places.

DERANGEMENTS

PROBLEM. Fifteen people check coats at a party and at the end they are handed back randomly. How likely is it that...

(a) Tim gets his coat back?
(b) Jeremy gets his coat back?
(c) Jeremy and Tim get their coats back?
(d) Jeremy and Tim get their coats back but no one else does?
(e) The members of the Beatles get the right set of coats back (maybe not in the right order)?
(f) Everyone gets their coat back?
(g) Exactly one person gets their coat back?
(h) Nobody gets their own coat back?
(i) At least one person gets their coat back?

Section 7.7
The Binomial Theorem

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

⋮

Pascal's Triangle

Theorem. The k^{th} entry in the n^{th} row of Pascal's triangle is $\binom{n}{k}$ for $n \geq 0$ and $0 \leq k \leq n$.

Note: The top row is considered to be row 0, and the leftmost entry is entry 0.

Proof. We use induction on n.

Base case $n=0$ ✓

Assume true for $n=k-1$.

Entries in row $k-1$ look like:

$$\cdots \binom{n-1}{k-1} \quad \binom{n-1}{k} \cdots$$

row n
entry k

So the k^{th} entry in row n is $\binom{n-1}{k-1}+\binom{n-1}{k}$.

Is this equal to $\binom{n}{k}$?

Yes – in choosing k objects, can either choose the n^{th} object or not. Use the addition rule. ∎

Pascal's Triangle

① What is 11^n for $n = 0, 1, 2, \ldots$?

$$11^0 = 1$$
$$11^1 = 11$$
$$11^2 = 121$$
$$11^3 = 1{,}331$$
$$\vdots$$

② What is the sum of the entries in the n^{th} row?

$$1 = 1$$
$$1 + 1 = 2$$
$$1 + 2 + 1 = 4$$
$$1 + 3 + 3 + 1 = 8$$
$$1 + 4 + 6 + 4 + 1 = 16$$
$$\vdots$$

The Binomial Theorem

Theorem. For any x and y and any natural number n, we have:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n} x^0 y^n$$

Proof. $(x+y)^n = (x+y)(x+y)\cdots(x+y)$

If we multiply out, we get an $x^{n-k}y^k$ term by choosing k of the x's. There are $\binom{n}{k}$ ways of doing this. ▨

The Binomial Theorem

Problem. Expand $(2x^3+y)^5$ and simplify.

Problem. Expand $\left(x-\frac{1}{x}\right)^6$ and simplify.

Problem. Find the coefficient of x^{15} in $\left(x^2-\frac{x}{3}\right)^{11}$.

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

plug in...	to prove...
$x=1,\ y=-1$	Inclusion-exclusion principle
$x=10,\ y=1$	n^{th} row of P's Δ $= 11^k$ ← HW
$x=1,\ y=1$	n^{th} row sum of P's Δ $= 2^n$
$x=\sqrt{2},\ y=-1$	$\sqrt{2}$ is irrational ← HW

The Inclusion-Exclusion Principle

Theorem. $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$

Proof: By the binomial theorem

$$0 = (1-1)^k = \binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k \binom{k}{k}$$

or $\binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k+1}\binom{k}{k} = 1$

Say an element of $\cup A_i$ is in k of the A_i.
The left hand side counts the number of times that element is counted by the inclusion-exclusion formula. So every element is counted once. $\blacksquare$

Row Sums in Pascal's Triangle

Theorem. The sum of the entries in the n^{th} row of Pascal's triangle is 2^n.

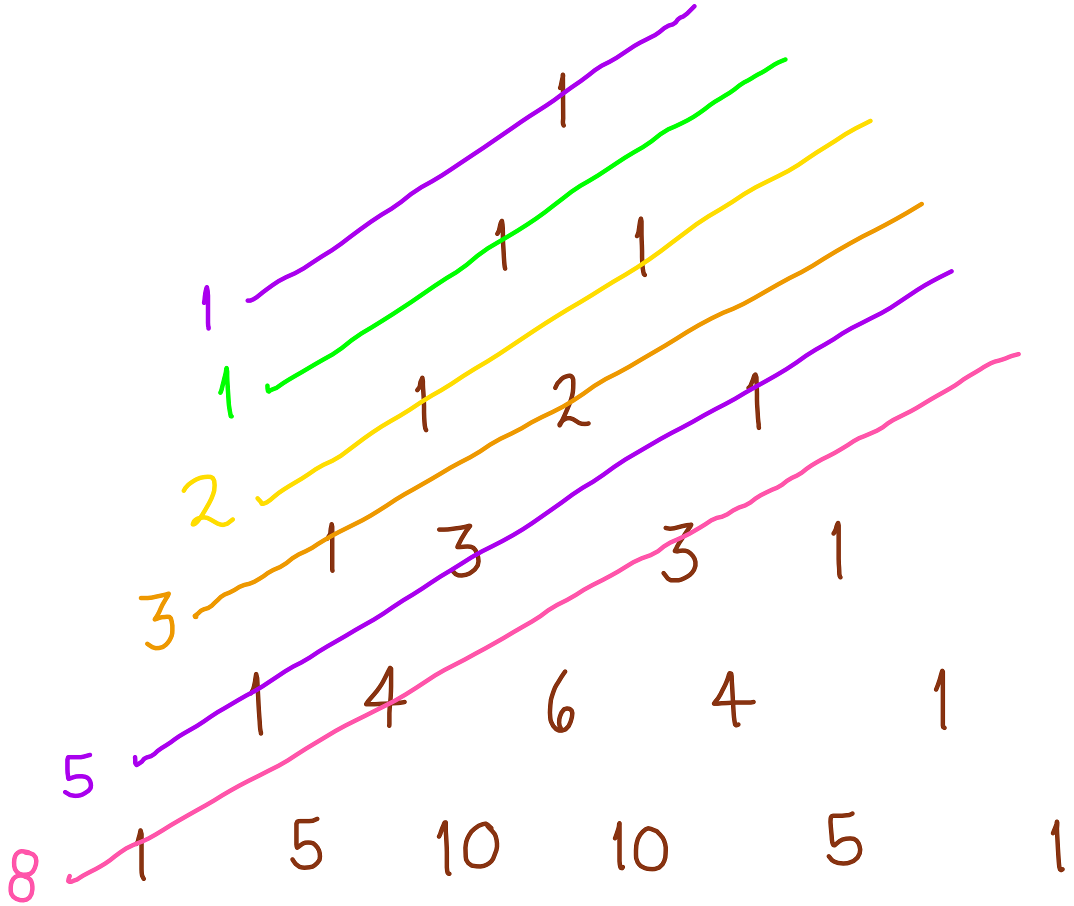
Proof. We have:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

But the $\binom{n}{k}$ are exactly the entries of the n^{th} row of Pascal's triangle. ▨

Another Proof. We know the number of subsets of a set with n elements is 2^n. The sum shown above just counts all subsets according to their size. ▨

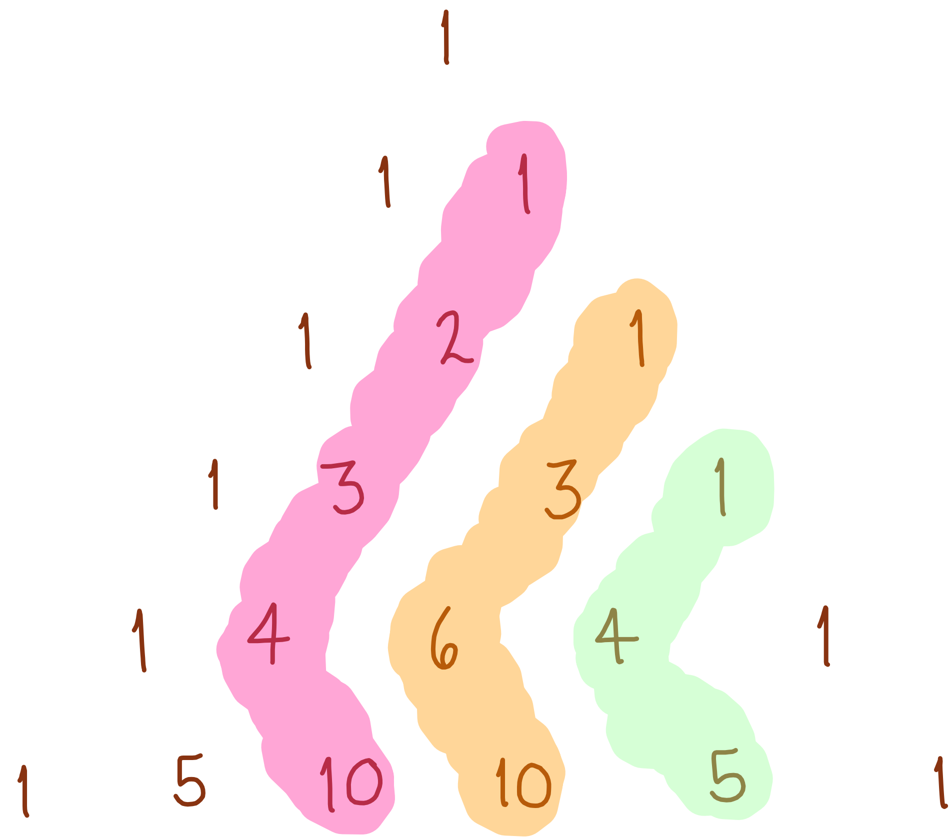
THE FIBONACCI NUMBERS IN PASCAL'S TRIANGLE



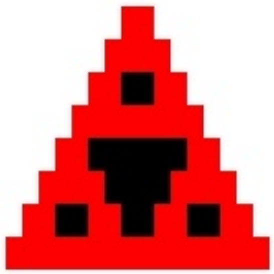
THEOREM. $$F_n = \begin{cases} \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{k}{k-1} & \text{if } n = 2k \\ \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{k}{k} & \text{if } n = 2k+1 \end{cases}$$

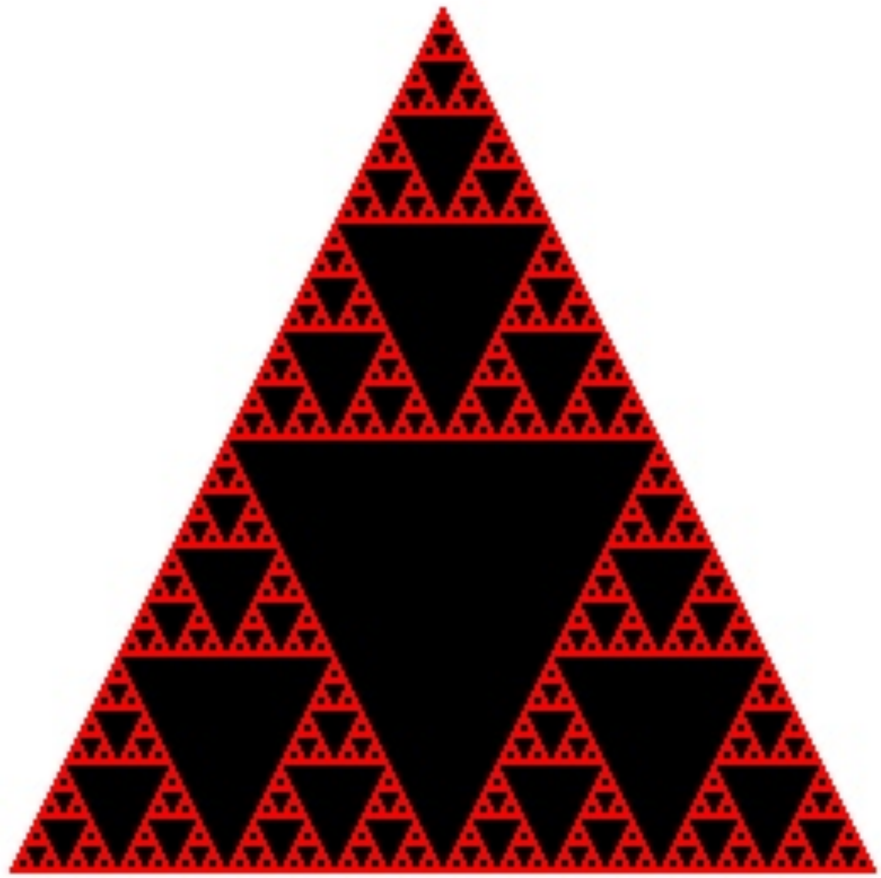
PROOF. Use induction. Hint: each pink = purple + orange.

The Hockey Stick Theorem



PASCAL'S TRIANGLE MOD 2





What about mod 3?