Chapter 9
Graphs

9.1 A Gentle Introduction

Four Problems

The Bridges of Konigsberg

Three House-Three Utility

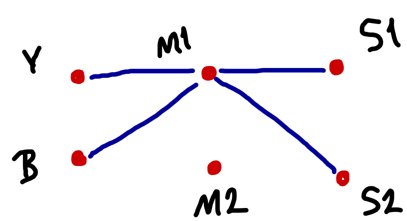
Four Color

Traveling Salesman

A Sample Problem

Among you, your buddy, two mothers, and two sisters, some people hug. There are no hugs between buddies, mothers, or sisters. The other 5 people tell you they all hugged different numbers of people. How many people did you hug?

Your buddy did not hug 4 people (otherwise noone hugged 0 people) → may assume your mother hugged 4



So your buddy's mom hugged 0.
If your buddy hugged 3, nobody hugged 1.
If your buddy hugged 1, nobody hugged 3.
→ your buddy hugged 2 → you hugged 2.

9.2 Definitions and Basic Properties

Graphs

A graph is a pair of sets V and E, where $V \neq \emptyset$ and each element of E is a pair of elements of V.

Write $G = G(V,E)$.

For us, graphs are finite, that is, $|V|$ is finite.

The elements of V and E are called vertices and edges.

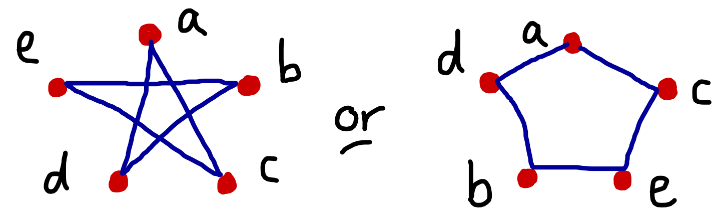
Example. V = Facebook users
E = Friendships

GRAPHS

We can represent graphs with pictures.

EXAMPLE. Consider the graph $G(V,E)$ where

$$V = \{a,b,c,d,e\}$$
$$E = \{\{d,b\},\{a,c\},\{e,b\},\{e,c\},\{d,a\}\}$$



Can describe a graph with a picture instead of set notation.

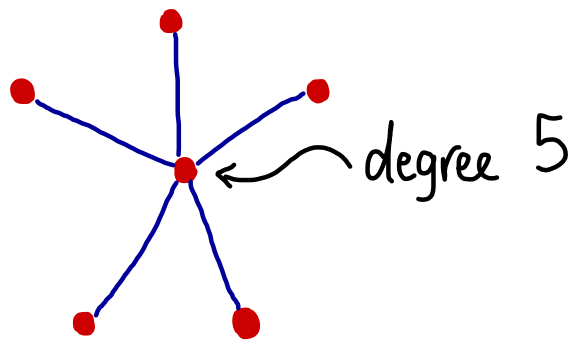
Could also write $E = \{db, ac, eb, ec, da\}$.

We say a is adjacent to c and d, and ac is incident to a and c.

DEGREES

The degree of a vertex v is the number of edges incident to v. Write $\deg v$.

If $\deg v = 0$, we say v is isolated.



Pseudographs

The following two phenomena are not allowed in a graph:

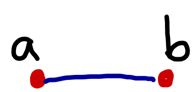
If we allow these, we get what is called a pseudograph.

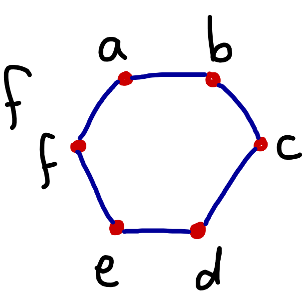
Pseudographs are harder to write down with set notation, so we usually describe them with a picture.

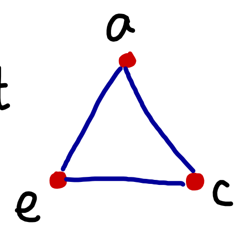
EXAMPLE. Vertices are web pages
Edges are links

Subgraphs

A subgraph of a graph $G(V,E)$ is a graph $G(V',E')$ where
$V' \subseteq V$ and
$E' \subseteq E$

EXAMPLE. 

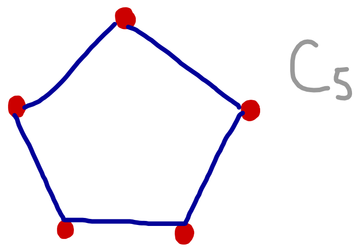
is a subgraph of 

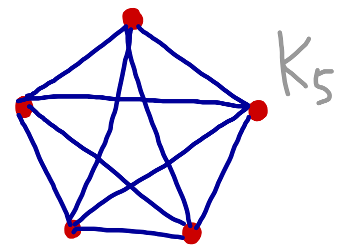
but 

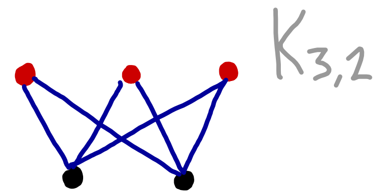
is not.

Also: Can delete any number of edges to get a subgraph.
Can delete any number of vertices (<u>and</u> all incident edges) to get a subgraph.

THREE SPECIAL FAMILIES







C_n

n-cycle

K_n

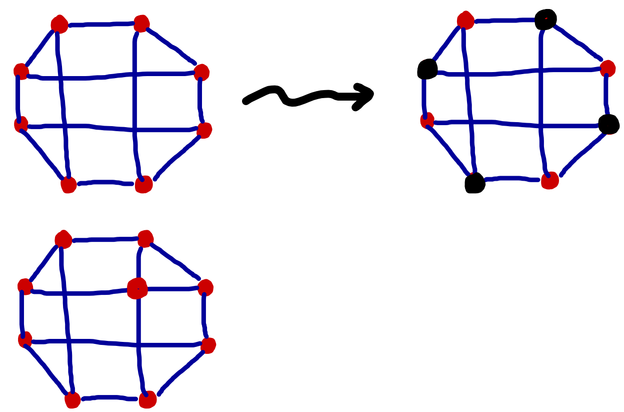
complete graph

$K_{m,n}$

complete bipartite graph

Bipartite Graphs

A bipartite graph is one whose vertex set can be partitioned into two sets V_1 and V_2 so that each edge joins an element of V_1 to an element of V_2.



Fact. A bipartite graph contains no triangles. More generally, a bipartite graph contains no odd cycles.

The Handshaking Lemma

Proposition. The sum of the degrees of the vertices of a pseudograph is an even number. Specifically:

$$\sum_{v \in V} \deg v = 2|E|$$

Leonhard Euler

Handshaking Lemma. The number of odd degree vertices of a pseudograph is even.

Proof. $\displaystyle\sum_{v \in V} \deg v = \sum_{v \text{ even}} \deg v + \sum_{v \text{ odd}} \deg v$

Revisit the hugging problem.

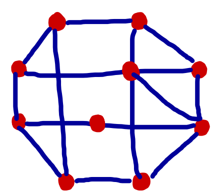
The Handshaking Lemma

Problem. A graph has 50 edges, 4 vertices of degree 2, 6 of degree 5, 8 of degree 4, all other vertices have degree 6. How many vertices does the graph have?

Problem. Out of 24 curling players, 78 pairs have played on the same team. Show that one has played on the same team as 7 others. Show that one has played on the same team with no more than 6 others.

Degree Sequence

Say $d_1, \ldots, d_n$ are the degrees of the vertices of a pseudograph, where $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $d_1, \ldots, d_n$ is the degree sequence of the pseudograph.

 $\rightsquigarrow$ 5, 4, 3, 3, 3, 3, 3, 3, 3, 2

9.3 Graph Isomorphism

Graph Isomorphism

Two graphs $G(V,E)$ and $G(V',E')$ are isomorphic if there is a bijection

$$V \to V'$$

that preserves adjacency and nonadjacency.

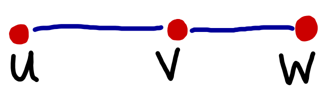
In other words, two graphs are isomorphic if there is a change of labels taking one to the other.

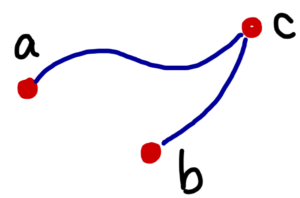
Example. $V = \{u, v, w\}$ $\quad$ $V' = \{a, b, c\}$ $\quad$ $V \to V'$

$E = \{uv, vw\}$ $\quad$ $E' = \{ac, cb\}$ $\quad$ $u \mapsto a$

$w \mapsto b$

$v \mapsto c$

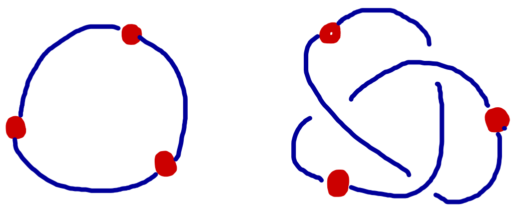




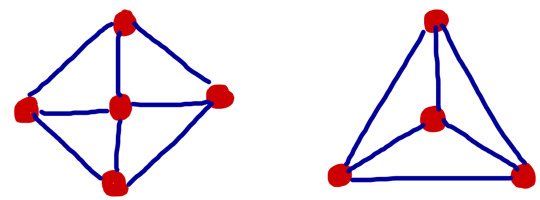
Graph Isomorphism

Which of the following pairs are isomorphic?

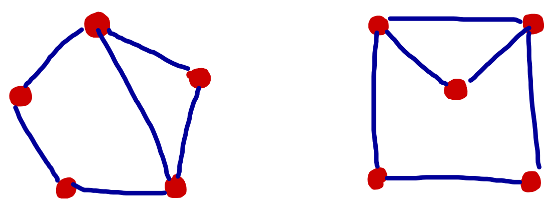
a)



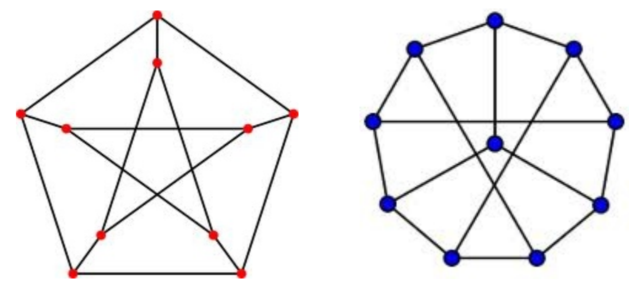
b)



c)



d)

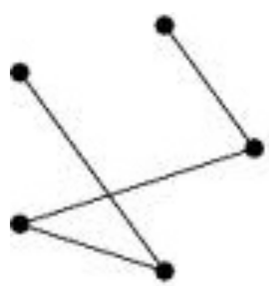


Invariants of Graphs

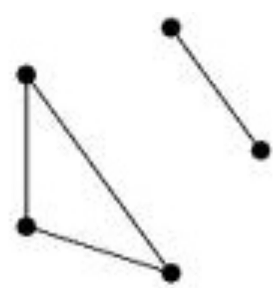
We can use the following "fingerprints" of graphs in order to tell if two graphs are different:

(i) Number of vertices
(ii) Number of edges
(iii) Degree sequence
etc.

It is possible for two graphs to have the same degree sequence and be nonisomorphic:



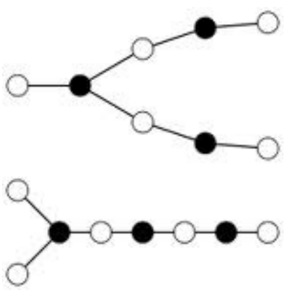
$\{2, 2, 2, 1, 1\}$

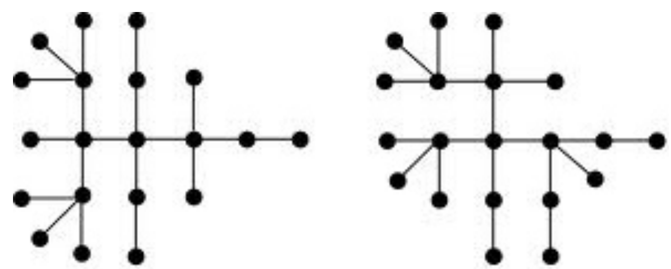


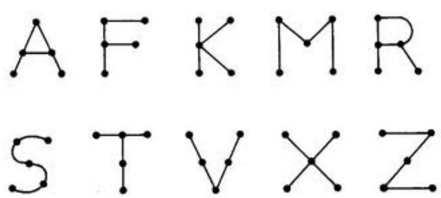
$\{2, 2, 2, 1, 1\}$

EXAMPLES

Which of the following graphs are isomorphic?

a) 

b) 

c) 

d)