Chapter 13
Planar Graphs and Colorings

13.1 Planar Graphs

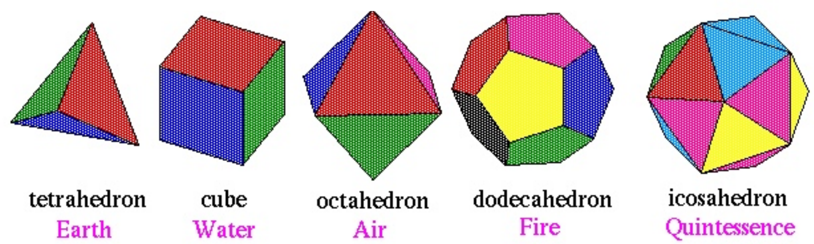
Planar Graphs

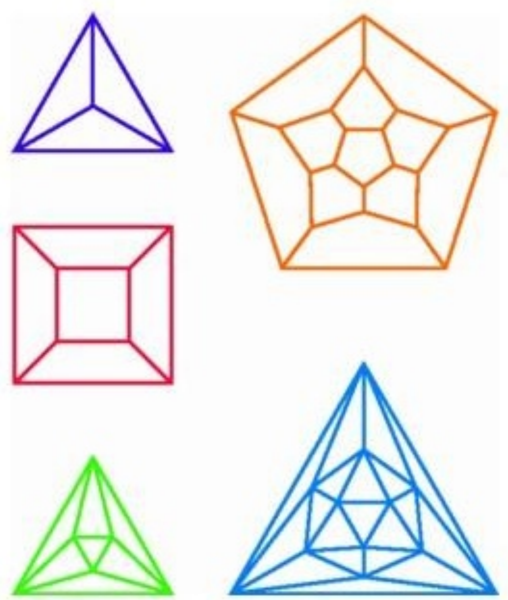
A graph is *planar* if it can be drawn in the plane so that no two edges cross.

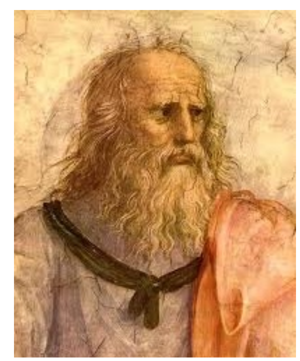
The Three House - Three Utility Problem asks whether or not $K_{3,3}$ is planar.

Platonic Solids

One collection of interesting planar graphs comes from the five Platonic solids:



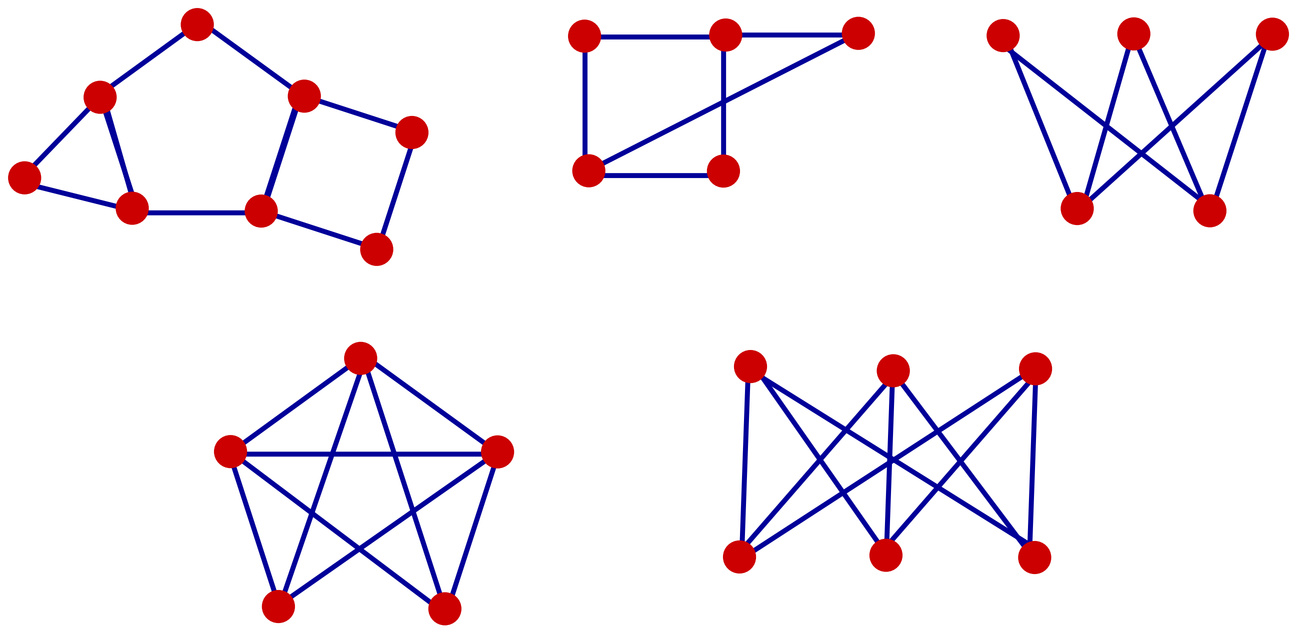




Plato

Planar Graphs

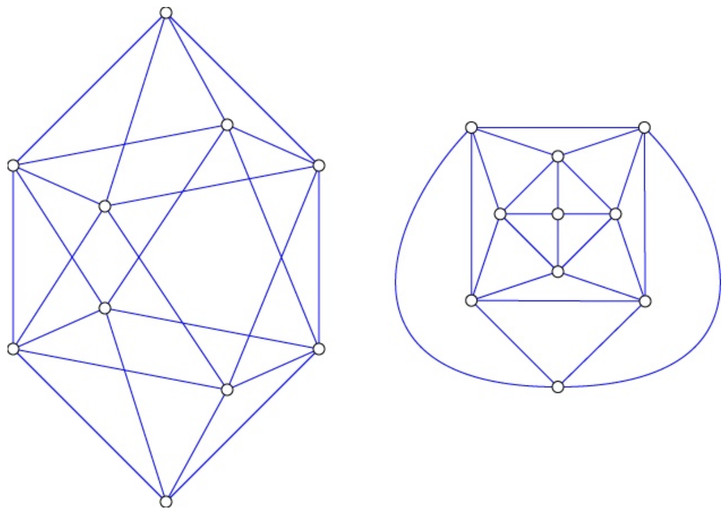
Which of the following graphs are planar?



Note: First translate each graph from a picture of a graph to an abstract graph.

Planar Graphs

To show that a graph is planar, you just need to draw it in the plane with no crossings:

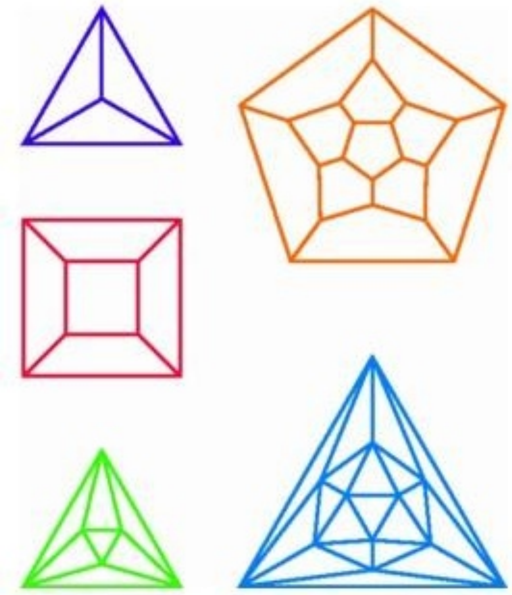


But how do we show a graph is <u>not</u> planar? For example, what about $K_{3,3}$? Is it possible to try all possible drawings? How many ways are there to draw $K_{2,2}$ or $K_{3,2}$ without crossings? Is there a better way?

Vertices, Edges, and Faces

A planar drawing of a planar graph divides the plane into distinct regions, or faces.

	vertices	edges	faces
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			



What is the pattern?

Euler's Theorem

Theorem. Any planar drawing of a graph with V vertices, E edges, and F faces satisfies

$$V - E + F = 2$$

In 1988, the Mathematical Intelligencer ran a survey. It was decided that the 5 most beautiful results in mathematics were:

(i) Euler's identity $e^{ix} = \cos x + i \sin x$
(ii) Euler's polyhedral formula $V - E + F = 2$
(iii) Euclid's proof of the infinitude of the primes
(iv) Euclid's proof that there are only 5 regular solids
(v) Euler's summation $\sum 1/n^2 = \pi^2/6$

Euler's Theorem

Theorem. Any planar drawing of a connected graph with V vertices, E edges, and F faces satisfies
$$V - E + F = 2$$

Proof. Induction on E.
Base case: $E = 0$ $\quad 1 - 0 + 1 = 2$ ✓
Assume the theorem is true for graphs with $E-1$ edges.
Case 1: G is a tree.
$\leadsto E = V-1,\ F = 1 \leadsto V - E + F = V - (V-1) + 1 = 2.$
Case 2: G is not a tree
$\leadsto G$ has a cycle.
Let e be an edge of G in some cycle.
Then $G-e$ has V vertices, $E-1$ edges, $F-1$ faces.
By induction $V - (E-1) + (F-1) = 2$
$\leadsto V - E + F = 2$ $\quad \blacksquare$

Platonic Solids

A Platonic solid is a 3-dimensional solid with polygonal faces, and satisfying:

(i) The faces are regular and congruent.
(ii) The same number of faces meet at each vertex.
(iii) The line connecting any two points on the solid is contained in the solid.

Theorem. There are exactly 5 Platonic solids.

Proof. Say we have a Platonic solid whose faces are n-gons, m at a vertex. $\rightsquigarrow$ Get a planar graph with

$$E = nF/2 \quad V = nF/m \qquad V - E + F = 2$$

$$\rightsquigarrow \quad 2E/m - E + 2E/n = 2$$

$$1/m + 1/n = 1/2 + 1/E$$

Notice $m, n \geq 3$. But by the above equation, m & n can't both be greater than 3... ▨

$K_{3,3}$ is not Planar

Theorem. $K_{3,3}$ is not planar.

Proof. Suppose it is planar with F faces.
$\rightsquigarrow V - E + F = 6 - 9 + F = 2 \rightsquigarrow F = 5.$
Let N = sum of boundary edges of each region.
Note $N \leq 2E = 18$ since each edge is used at most twice.
But each face must have at least 4 sides, since $K_{3,3}$ has
no triangles $\rightsquigarrow N \geq 5 \cdot 4 = 20$. Contradiction. $\blacksquare$

K_5 IS NOT PLANAR

THEOREM. If a planar graph has V vertices and E edges, then $E \le 3V-6$.

PROOF. We may assume G is connected. Why?
If $V=3$ then $E \le 3$ ✓
Now assume $V \ge 4$, $E \ge 3$.
Let N be as before. Again $N \le 2E$
Also $N \ge 3F$ since each face has at least 3 sides.
$\rightsquigarrow 3F \le 2E$

$$6 = 3V-3E+3F \le 3V-3E+2E = 3V-E$$

$\rightsquigarrow E \le 3V-6$ ∎

COROLLARY. K_5 is not planar.

PROOF. $V=5$, $E=\binom{5}{2}=10 > 3V-6=9$. ∎

Degrees

Theorem. Every planar graph has at least one vertex whose degree is less than 6.

Proof. Say all degrees are ≥ 6.

$2E = \text{sum of degrees} \geq 6V$

$\rightsquigarrow E \geq 3V > 3V-6$. $\blacksquare$

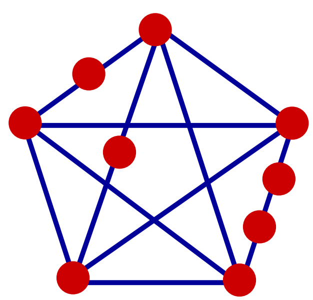
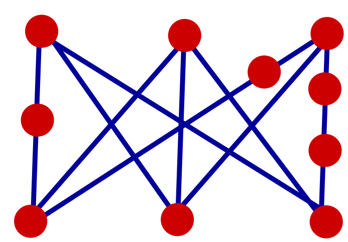
More Nonplanar Graphs

So far, we know K_5 and $K_{3\,3}$ are not planar.
It follows that K_n is not planar for $n \geqslant 5$,
$K_{m,n}$ is not planar for $m,n \geqslant 3$.

More generally:

Proposition. Any graph that contains K_5 or $K_{3,3}$ as a subgraph is not planar.

Note also any subdivision of K_5 or K_3 is nonplanar:

Proposition. Any graph that contains a subdivision of K_5 or $K_{3,3}$ as a subgraph is not planar.

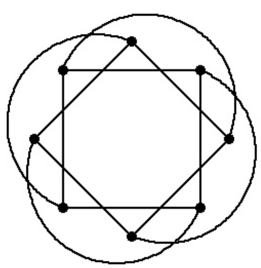
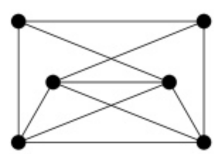
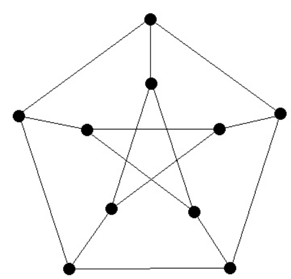
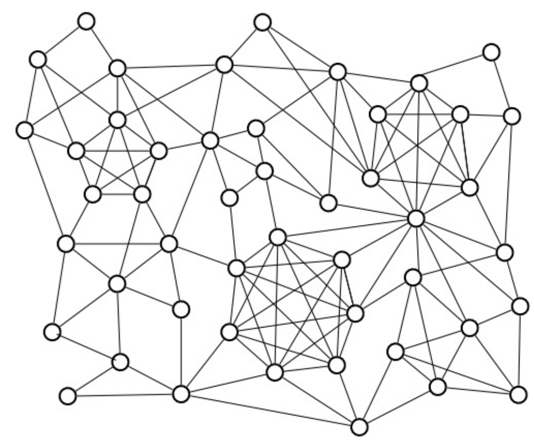
Kuratowski's Theorem

Amazingly, the converse is also true:

Theorem. A graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.

Proof. See web site.

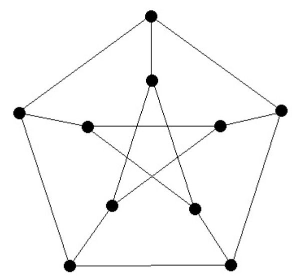
Which of the following graphs are planar?

Wagner's Theorem

A graph H is a minor of a graph G if H is obtained from G by taking a subgraph and collapsing some edges.

THEOREM. A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.



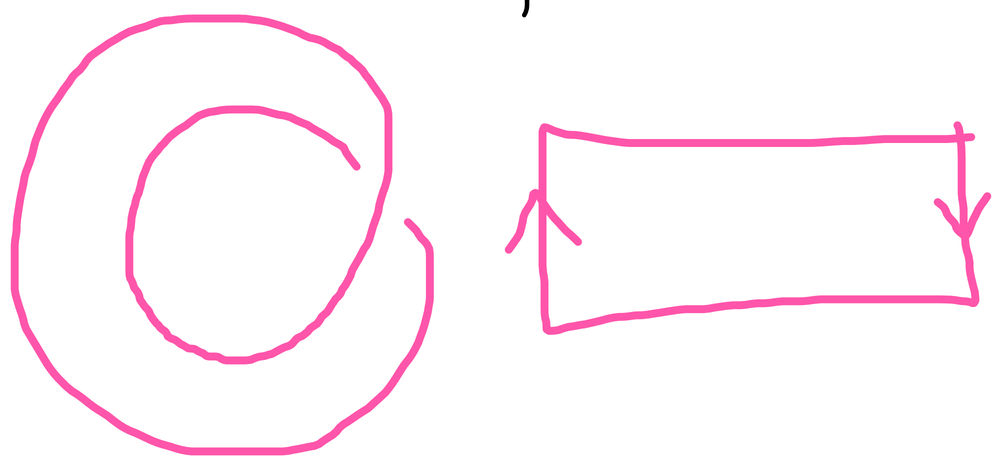
FÀRY'S THEOREM

THEOREM. Every planar graph can be drawn in the plane using only straight lines.

The proof uses the art gallery theorem...

OTHER SURFACES

What are the largest m, n so K_n and $K_{m,n}$ can be drawn without crossings on a Möbius strip



or a torus?