5.1 Mathematical Induction

THE PRINCIPLE OF MATHEMATICAL INDUCTION

Say we have a mathematical statement that depends on a natural number n. Suppose:

1. The statement is true for $n = n_0$.
2. Whenever the statement is true for $n = k$, it is true for $n = k+1$.

Then the statement is true for all $n \geq n_0$.

EXAMPLE

PROPOSITION: For $n \geq 0$ $\quad (1+1/2)^n \geq 1+\frac{n}{2}$

PROOF: First we check the base case $n=0$:

$$1 = (1+1/2)^0 \geq 1 + 0/2 = 1$$

Next, we assume the proposition is true for $n=k$:

$$(1+1/2)^k \geq 1 + k/2$$

Using the assumption, we prove the proposition for $n=k+1$:

$$\begin{aligned}(1+1/2)^{k+1} = (1+1/2)(1+1/2)^k &\geq (1+1/2)(1+k/2) \\ &= 1 + k/2 + 1/2 + k/4 \\ &\geq 1 + (k/2 + 1/2) \\ &= 1 + (k+1)/2\end{aligned}$$

By the principle of mathematical induction, the proposition is proven. $\blacksquare$

THE STRONG FORM OF THE PRINCIPAL OF MATHEMATICAL INDUCTION

Say we have a mathematical statement that depends on a natural number n. Suppose that

1. The statement is true for $n = n_0$
2. Whenever the statement is true for all natural numbers in the interval $[n_0, k]$, then it is also true for $n = k+1$.

Then the statement is true for all $n \geq n_0$.

Note: It may be that more than one base case is needed! The number of base cases needed is dictated by the inductive argument.

EXAMPLE

PROPOSITION: Every natural number $n \geq 2$ is a product of prime numbers.

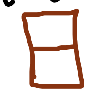
PROOF: The base case $n=2$ is obviously true.
Now, assume that every natural number n in $[2, k-1]$ is a product of prime numbers. We must show that k is a product of prime numbers.

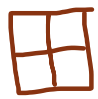
First, if k is prime, there is nothing to do. On the other hand, if k is not prime, it is equal to a product $k = mn$, where $2 \leq m, n < k$. By our inductive hypothesis, both m and n are products of prime numbers. Therefore, k is itself a product of prime numbers. ∎

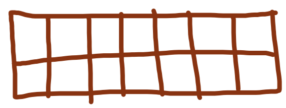
EXAMPLE

PROPOSITION: The number of ways of breaking a $2 \times n$ candy bar into 2×1 bars is F_{n+1}.

PROOF: First we check the base cases $n = 1$ and $n = 2$:

 only 1 way, and $F_2 = 1$.

 two ways, and $F_3 = 2$

We assume the proposition is true for $1, \ldots, k-1$, where $k \geq 3$. We must now prove the proposition for $n = k$: 

There are two ways to break off the end: one vertical piece, or two 2×1 horizontal pieces.

In the 1st case we get a $2 \times (k-1)$ bar $\rightsquigarrow F_k$ ways.

2nd case $\rightsquigarrow 2 \times (k-2)$ bar $\rightsquigarrow F_{k-1}$ ways

In total, $F_{k-1} + F_k = F_{k+1}$ ways to break the $2 \times k$ bar.

By strong induction, the proposition is proven.

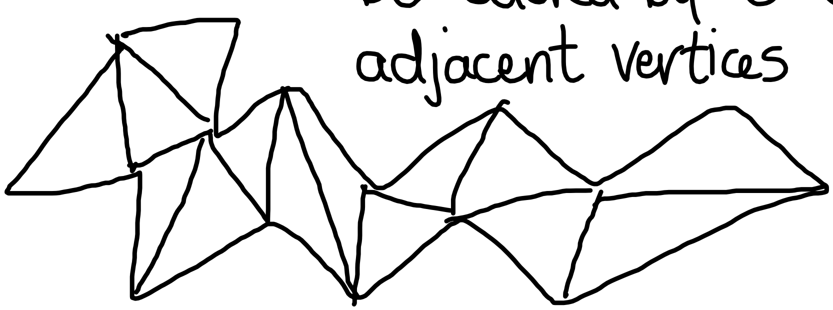
MORE EXAMPLES

Consider the sequence $a_1 = 1,\ a_2 = 2,\ a_3 = 3$
$a_k = a_{k-1} + a_{k-2} + a_{k-3}$ for $k \geq 4$.

PROPOSITION: $a_n < 2^n$ for all $n > 0$.

PROPOSITION: In a regular n-gon, one can draw at most $n-3$ diagonals that do not cross.

PROPOSITION: The vertices of a triangulated n-gon can always be colored by 3 colors so that no two adjacent vertices have the same color.



Section 5.2
Recurrence Relations

RECURRENCE RELATIONS

A recurrence relation for a sequence $(a_n)_{n=0}^{\infty}$ is an equation expressing each term a_n in terms of its predecessors $a_1, \ldots, a_{n-1}$.

If some a_i are given specific values, those are called initial conditions.

A first example:

$$a_n = a_{n-1} + 3, \quad a_0 = 0.$$

Solution: $a_n = 3n$

Like a differential eqn:

$$f'(x) = 3 \qquad f(0) = 0$$

Solution: $f(x) = 3x$

EXAMPLES OF RECURRENCE RELATIONS

	Closed form	Recursive form
Exponentials	$a_n = 2^n$	$a_n = 2a_{n-1}$, $a_0 = 1$
Factorials	$a_n = n!$	$a_n = na_{n-1}$, $a_0 = 1$
Arithmetic seq.	$a_n = dn + b$	$a_n = a_{n-1} + d$, $a_0 = b$
Geometric seq.	$a_n = cr^n$	$a_n = ra_{n-1}$, $a_0 = c$

↳ e.g. Money market account:
Put in \$500, collect 7% annually

$$a_n = 500(1.07)^n$$

MORE EXAMPLES

Annuity: Deposit \$200/yr, get 7% interest/year

$a_n = 1.07 \cdot a_{n-1} + 200$

closed form?

Fibonacci numbers: $a_0 = 0,\ a_1 = 1$

$a_n = a_{n-1} + a_{n-2}$

closed form?

Wilhelm Ackermann

Ackermann function:
(i) $A(n,0) = A(n-1,1) \quad n = 1,2,\ldots$
(ii) $A(n,k) = A(n-1, A(n,k-1)) \quad n,k = 1,2,\ldots$
(iii) $A(0,k) = k+1 \quad k = 0,1,\ldots$

Very hard to compute:

$A(0,0) = 1,\ A(1,1) = 3,\ A(2,2) = 7,\ A(3,3) = 61$

MORE EXAMPLES

Annuity: Deposit \$200/yr, get 7% interest/year

$$a_n = 1.07 \cdot a_{n-1} + 200$$

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Fibonacci numbers: $a_0 = 0,\ a_1 = 1$

$$a_n = a_{n-1} + a_{n-2}$$

closed form?

Ackermann Function:
(i) $A(n,0) = A(n-1,1) \quad n = 1,2,\ldots$
(ii) $A(n,k) = A(n-1, A(n,k-1)) \quad n,k = 1,2,\ldots$
(iii) $A(0,k) = k+1 \quad k = 0,1,\ldots$

Very hard to compute:

$A(0,0)=1,\ A(1,1)=3,\ A(2,2)=7,\ A(3,3)=61$

over 10^{19199} digits $\longrightarrow$ $A(4,4) = 2^{2^{2^{2^{2^{2^{2}}}}}} - 3$

universe has $\sim 10^{80}$ elementary particles

SOLVING RECURRENCE RELATIONS

A solution to a recurrence relation is an explicit formula for the sequence.

Example: Consider the arithmetic sequence

$$a_0 = -2, \quad a_n = a_{n-1} + 5.$$

Solution:

$$a_n = 5n - 2$$

More generally: $a_0 = b, \; a_n = a_{n-1} + m$

Solution:

$$a_n = mn + b$$

Can prove by induction.

SOLVING RECURRENCE RELATIONS

Example: Consider

$$a_0 = 7, \ a_n = -3a_{n-1}$$

Solution:

$$a_n = 7(-3)^n$$

More generally: Consider

$$a_0 = c, \ a_n = ra_{n-1}$$

Solution:

$$a_n = ra_{n-1}$$

A MORE INTERESTING EXAMPLE

Example: Solve the recurrence relation

$$a_0 = 0, \; a_n = 2a_{n-1} + 1$$

Note: this formula describes the number of moves in our solution to the Towers of Hanoi puzzle.

Let's find the first few terms:

$$a_0 = 0$$
$$a_1 = 1$$
$$a_2 = 3$$
$$a_3 = 7$$
$$a_4 = 15$$
$$a_5 = 31$$

Guess: $a_n = 2^n - 1$

VERIFYING OUR GUESS

PROPOSITION: The solution to

$$a_0 = 0, \quad a_n = 2a_{n-1} + 1$$

is $a_n = 2^n - 1$.

PROOF: We proceed by induction on n.

Base case $n = 0$: $a_0 = 0 = 2^0 - 1$ ✓

Assume the proposition is true for $n = k$:

$$a_k = 2^k - 1$$

Using the assumption, we show the proposition is true for $n = k+1$:

$$\begin{aligned} a_{k+1} &= 2a_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

✓

$\blacksquare$

MORE COMPLICATED RECURSION RELATIONS

What about $a_0 = 1, \quad a_n = 2a_{n-1} + 3$?

or $a_0 = 1, \; a_1 = 2, \; a_n = 2a_{n-1} + 3a_{n-2}$?