13.2 Coloring Graphs

The Four Color Problem

Show that, given any map in the plane, you can color it with four colors so that adjacent regions have different colors.

Notes. (i) Each region must be a connected "blob".
(ii) "Adjacent" means the regions meet in a segment (not just a corner).

Why are these caveats needed?

Francis Guthrie

Is there a map that really requires 4 colors?

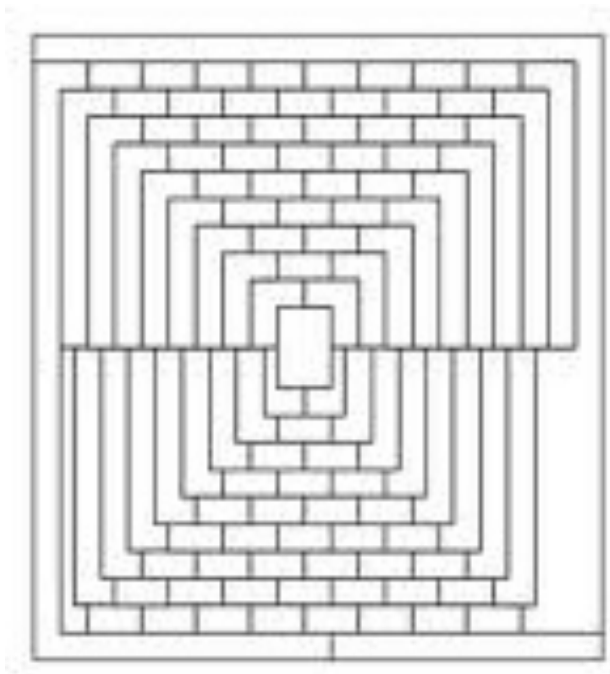
The Four Color Problem

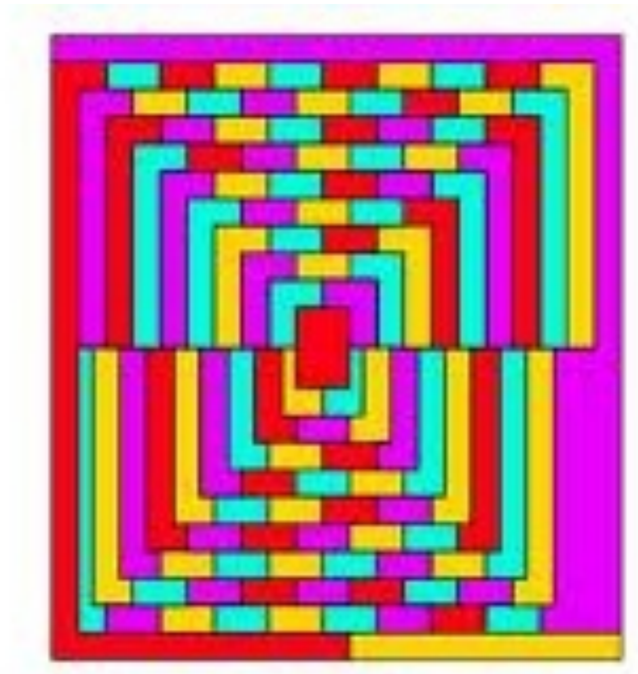
How many colors are needed?

Hint: Look at Nevada.

The Four Color Problem

How many colors are needed?





For more challenges: nikoli.com

The Four Color Problem

First posed in 1852 by Guthrie. Many tried to solve it. Alfred Kempe (1879) and Pether Guthrie Tait (1880) both gave solutions that stood for 11 years.

Lewis Carroll wrote about it:

"A is to draw a fictitious map divided into counties.

B is to color it (or rather mark the counties with names of colours) using as few colours as possible.

Two adjacent counties must have different colours.

A's object is to force B to use as many colours as possible. How many can he force B to use?"

The problem was solved in 1976 by Appel and Haken. It was the first major theorem proven in large part by computer.

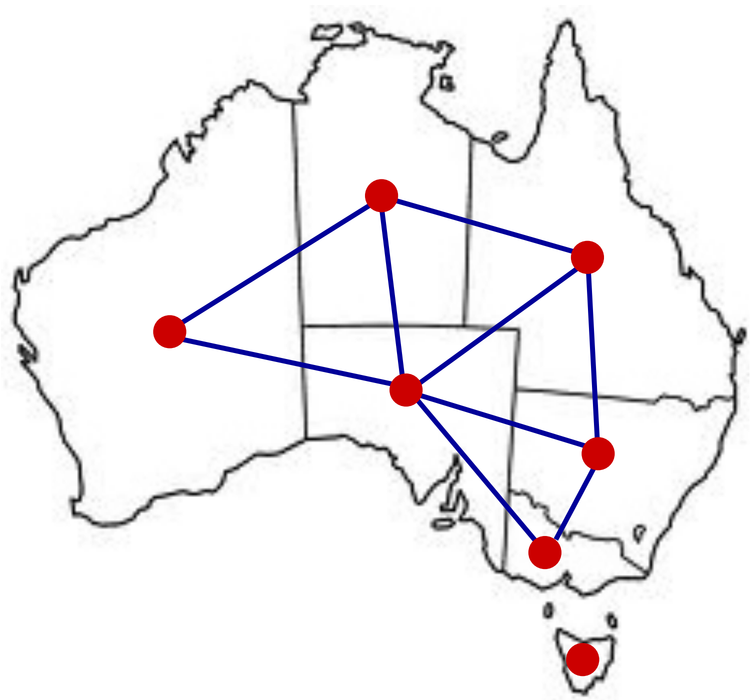
The proof has recently been simplified by Robin Thomas (GaTech) and his collaborators (still using computers).

Back to Graphs

Given a map, we get a graph G(V,E) where

V = {regions}

E = {pairs of adjacent regions}



If the map is planar, then the graph is planar.

Coloring the map corresponds to coloring the vertices of the graph so that adjacent vertices have different colors.

Graph Coloring

A coloring of a graph is an assignment of colors to each of the vertices so that adjacent vertices have different colors.

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed for a coloring of G.

Fact. $1 \leq \chi(G) \leq |V|$

Fact. If G is isomorphic to H, then $\chi(G) = \chi(G)$.

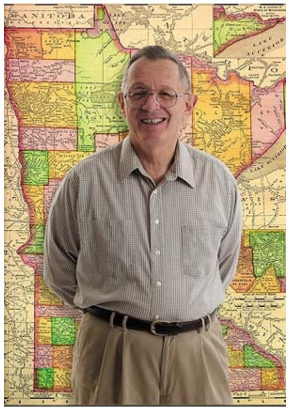
Fact. $\chi(K_n) = n$, $\chi(K_{m,n}) = 2$, and $\chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd.} \end{cases}$

Fact. If H is a subgraph of G then $\chi(H) \leq \chi(G)$

Fact. If G has a coloring with n colors, then $\chi(G) \leq n$.

The Four Color Theorem

Theorem. If G is planar, then $\chi(G) \leq 4$.



Kenneth Appel

Wolfgang Haken

Note: There is still no polynomial time algorithm for finding a coloring with 4 colors.

Applications

1. Sudoku. A vertex for each little square.
An edge for two squares in same row, col, or 3×3 sqr.

2. Radio Frequencies. A vertex for each radio station.
An edge between stations that are near eachother.

3. Scheduling. Example: Say there are 10 students taking

1. Physics, Math, IE
2. Physics, Econ, Geology
3. Geology, Business
4. Stat, Econ
5. Math, Business
6. Physics, Geology
7. Business, Stat
8. Math, Geology
9. Physics, Comp Sci, Stat
10. Physics, Econ, Comp Sci

What is the minimum number of final exam periods needed?

Six Colors Suffice

Proposition. If G is a planar graph then $\chi(G) \leq 6$.

Proof. Induction on the number of vertices.

Base case: one vertex ✓

Assume the proposition is true for planar graphs with $n-1$ vertices.

Let G be a planar graph with $n \geq 2$ vertices.

Recall that any planar graph has a vertex v of degree ≤ 5.

By induction, we can color $G - v$ with 6 colors.

Then color v differently from its neighbors using the sixth color. ∎

Degrees and Colors

Proposition. For any graph G:

$$\chi(G) \leq (\text{largest degree of a vertex of } G) + 1$$

Proof. Same as above.

Computing χ

To show that $\chi(G) = n$, we generally have to show two things:

① $\chi(G) \leq n$

Some possible reasons:

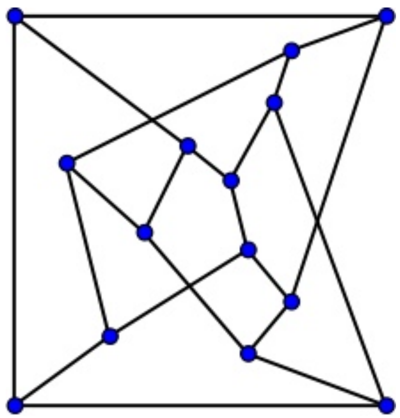
- G has n vertices
- G is bipartite
- G is planar
- Largest vertex degree is $n+1$
- We know an explicit coloring with n vertices.

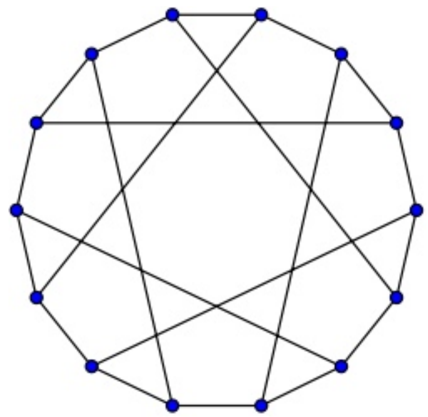
② $\chi(G) \geq n$

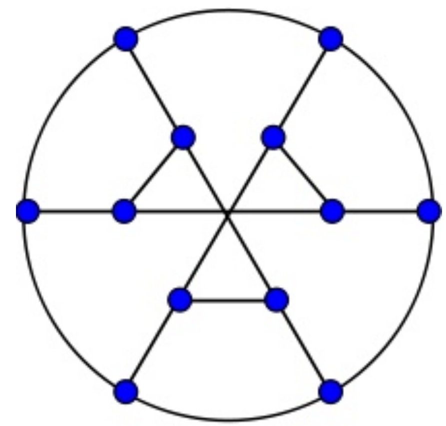
Some possible reasons:

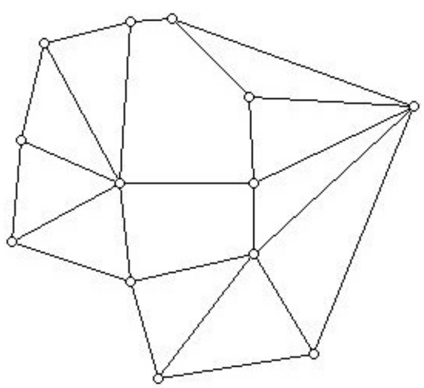
- G contains H and $\chi(H) = n$
- G contains H with $\chi(H) = n-1$ and a vertex adjacent to each vertex of H (cf. Nevada)

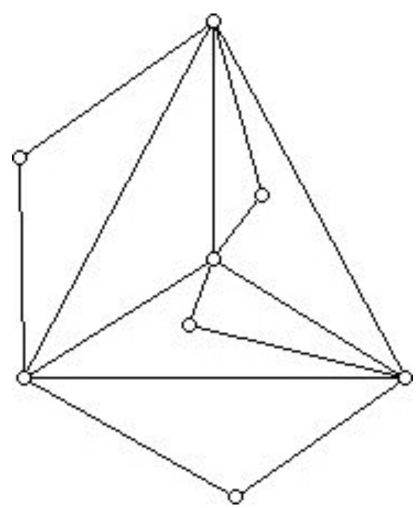
More Coloring Problems

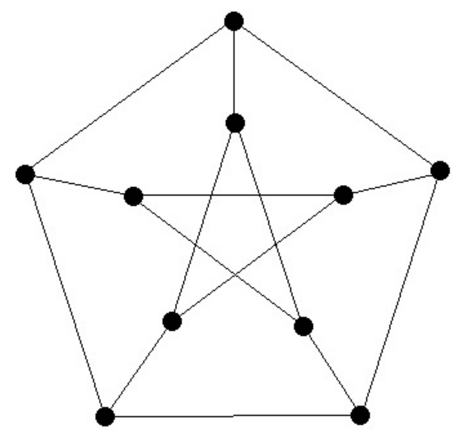










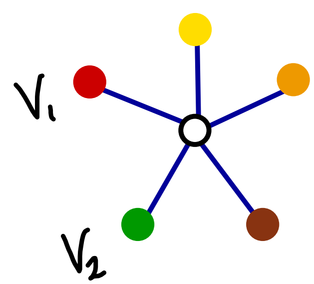


Five Colors Suffice

Percy Heawood

Theorem. If G is a planar graph, then $\chi(G) \leq 5$.

Proof. Induction on # vertices again.
Say G is a planar graph with n vertices.
As before, delete a vertex v of degree ≤ 5. Color $G - v$ with 5 colors. Can we reinsert v?



Case 1. There is no path from v_1 to v_2 using only red and green vertices.
In this case, starting at v_1, swap red and green.
Then color v red.

Case 2. There is such a path. Similar. ∎