2 Solving Linear Systems

2.1 Echelon Form of a Matrix

Solving Linear Systems

Solve $\begin{matrix} 3x+3y=9 \\ 3x+y=7 \end{matrix} \xrightarrow{\text{subtract}} \begin{matrix} 3x+3y=9 \\ 2y=2 \end{matrix} \xrightarrow{\text{solve}} y=2 \xrightarrow[\text{sub}]{\text{back}} x=2$

Goal: eliminate variables

Can compactify this information using matrices:

$$\begin{pmatrix} 3 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

Goal: create zeros.

Simplifying the matrices $\leftrightarrow$ simplifying the equations

Row Echelon Form

An $m \times n$ matrix is in reduced row echelon form if:

1. Any zero rows are at the bottom.
2. The first nonzero entry of a row is 1 (called a "leading 1")
3. A leading 1 lies to the right of all leading 1's above it.
4. If a column has a leading 1, all other entries in that column are zero.

4 means reduced.

Example.

$$\begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{aligned} a + 4b &= 0 \\ c &= 2 \\ d &= 4 \end{aligned}$$

It is easy to solve the corresponding linear system.

Row Echelon Form

échelon = rung of a ladder.

Originally used to describe a formation of troops:

Row Echelon Form

Which matrices are in reduced row echelon form?

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If not, which criteria do they fail?

If a matrix is not in reduced row echelon form, can we make it so?

Row Operations

An elementary row operation on a matrix is any of:

Type I: interchange any two rows $r_i \leftrightarrow r_j$

Type II: multiply a row by a number $kr_i \to r_i$

Type III: add a multiple of one row to another $kr_i + r_j \to r_j$

Examples:

$$\begin{pmatrix} 4 & 5 & 7 & 9 \\ 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{3r_2 \to r_2} \begin{pmatrix} 4 & 5 & 7 & 9 \\ 3 & 6 & 9 & 12 \\ 2 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 4 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

$$\xrightarrow{-2r_2 + r_1 \to r_1} \begin{pmatrix} 0 & 1 & 3 & 5 \\ 2 & 2 & 2 & 2 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

Row Equivalence

Two $m \times n$ matrices are row equivalent if one can be obtained from the other by a sequence of elementary row operations.

The matrices on the last page are all row equivalent.

Row equivalence is an equivalence relation:

(i) A is row equivalent to A.

(ii) If A is row equivalent to B then B is row equivalent to A.

(iii) If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

Reducing Matrices

Theorem. Every nonzero $m \times n$ matrix is row equivalent to a unique matrix in reduced row echelon form.

Recipe. Look at the first column with a nonzero entry.
Make that entry a 1 (Type II).
Move that 1 to the first row without a leading 1 (Type I)
Make all other entries in that column 0 (Type III).
Repeat: Find first column with nonzero entry below the last leading 1, ...

Examples. Find the reduced row echelon form:

$$\begin{pmatrix} 0 & 2 & 8 & -7 \\ 2 & -2 & 4 & 0 \\ -3 & 4 & -2 & 5 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 0 & 5 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{pmatrix}$$

2.2 Solving Linear Systems

Augmented Matrices

We solved $\begin{aligned} 3x+3y&=9 \\ 3x+y&=7 \end{aligned}$ via row operations on $\begin{pmatrix} 3 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$

We can go one step further and drop the x and y

$\rightsquigarrow$ augmented matrix $\left(\begin{array}{cc|c} 3 & 3 & 9 \\ 3 & 1 & 7 \end{array}\right)$

THEOREM. If two augmented matrices differ by row operations, then the corresponding linear systems have the same solutions.

Homogeneous Systems

If the last column of an augmented matrix is the zero vector, the linear system is called homogeneous.

Arbitrary linear systems: $Ax=b$, e.g. $\begin{aligned} 3x+3y&=9 \\ 3x+y&=7 \end{aligned}$

Homogeneous linear systems: $Ax=0$, e.g. $\begin{aligned} 3x+3y&=0 \\ x+2y&=0 \end{aligned}$

In the homogeneous case, we can ignore the last column.

Homogeneous systems always have at least one solution.

Say A is an $n \times n$ matrix. The homogeneous system $Ax=0$ has a nonzero solution if and only if $\det(A)=0$ if and only if the last row of the row echelon form of A is all 0.

How Many Solutions?

The solution set to a system of linear equations can be:

(i) the empty set (no solutions)
(ii) a point (one solution)
(iii) a line (infinitely many solutions)
(iv) a plane (infinitely man solutions)
etc.

Say we are solving $Ax=b$.

We can easily see which case we are in by putting $(A|b)$ in reduced row echelon form.

How Many Solutions?

1. $$\begin{aligned} x+y \quad &= 0 \\ z &= 0 \\ 0 &= 1 \end{aligned} \quad \longleftrightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

no solutions

2. $$\begin{aligned} x \quad &= 0 \\ y \quad &= 0 \\ z &= 1 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

one solution

3. $$\begin{aligned} x + \quad z &= 0 \\ y \quad &= 1 \\ 0 &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

∞ many solutions:

$y = 1$

$x = -z$

or: $(s, 1, -s)$

In general, variables that don't correspond to leading 1's are free.

How Many Solutions?

1. $x - 3y + z = 4$
   $2x - 8y + 8z = -2$
   $-6x + 3y - 15z = 9$

one solution:
$x = 3,\ y = -1,\ z = -2$

2. $2x - y + z = 1$
   $3x + 2y + 4z = 4$
   $-6x + 3y - 3z = 2$

no solutions (the first equation is almost a multiple of the third)

3. $x + y + z = 12$
   $3x - 2y + z = 11$
   $5x \quad + 3z = 35$

∞ many solutions:
$x = 7 - \frac{3}{5}z$
$y = 5 - \frac{2}{5}z$
or: $(7 - \frac{3}{5}s,\ 5 - \frac{2}{5}s,\ s)$

How Many Solutions?

4. $$\begin{aligned} 2x+4y+6z&=18\\ 4x+5y+6z&=24\\ 2x+7y+12z&=40 \end{aligned}$$

5. $$\begin{aligned} 2x+4y+6z&=18\\ 4x+5y+6z&=24\\ 3x+y-2z&=4 \end{aligned}$$

6. $$\begin{aligned} 2x+4y+6z&=18\\ 4x+5y+6z&=24\\ 2x+7y+12z&=30 \end{aligned}$$

7. $$\begin{aligned} x+2y+3z&=6\\ 2x-3y+2z&=14\\ 3x+y-z&=-2 \end{aligned}$$

More Variables Than Equations

Theorem. If a system of linear equations has more variables than equations, then there are either no solutions or infinitely many.

In particular, if the system is homogeneous, there are infinitely many.
More specifically, the number of free parameters is the number of variables minus the number of equations.

Examples.

$$\begin{aligned} x_1 + x_2 + 2x_3 \quad - \tfrac{5}{2}x_5 &= \tfrac{2}{3} \\ x_4 + \tfrac{1}{2}x_5 &= \tfrac{1}{2} \end{aligned} \rightsquigarrow x_2, x_3, x_5 \text{ are free}$$

$$\begin{aligned} x + 3y - 5z + w &= 4 \\ 2x + 5y - 2z + 4w &= 6 \end{aligned} \rightsquigarrow \begin{aligned} x &= -2 - 19z - 7w \\ y &= 2 + 8z + 2w \end{aligned}$$

Linear Transformations

A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function

$$T: \mathbb{R}^n \to \mathbb{R}^m$$

with: (i) $T(kv) = kT(v)$ for $v \in \mathbb{R}^n$, $k \in \mathbb{R}$

(ii) $T(v+w) = T(v) + T(w)$ for $v, w \in \mathbb{R}^n$

$$\left\{ \begin{array}{c} \text{linear transformations} \\ \mathbb{R}^n \to \mathbb{R}^m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} m \times n \\ \text{matrices} \end{array} \right\}$$

- Given a linear transformation T we get a matrix whose column vectors are $T(e_1), \ldots, T(e_n)$
- Given a matrix M, we get a linear transformation $T(v) = Mv$

example: $T(x,y) = (5x - 3y,\ x) \longleftrightarrow \begin{pmatrix} 5 & -3 \\ 1 & 0 \end{pmatrix}$

Linear Transformations

The *range* of a function $f: A \to B$ is $\{b \in B : b = f(a) \text{ for some } a\}$.

Problem. What is the range of the linear transformation associated to the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{pmatrix} ?$$

In other words, what are conditions on a, b, c so that

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

for some x, y, z?

Make the augmented matrix and try to solve.

Homogeneous versus Nonhomogeneous

THEOREM. The set of solutions to $Ax=b$, $b\neq 0$, is

$$\{x_p + x_h\},$$

where x_p is any particular solution to $Ax=b$ and x_h ranges over all solutions to $Ax=0$.

Compare with the case. In fact, prove the corresponding theorem about recurrence relations using this theorem.