Chapter 7

7.1 Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

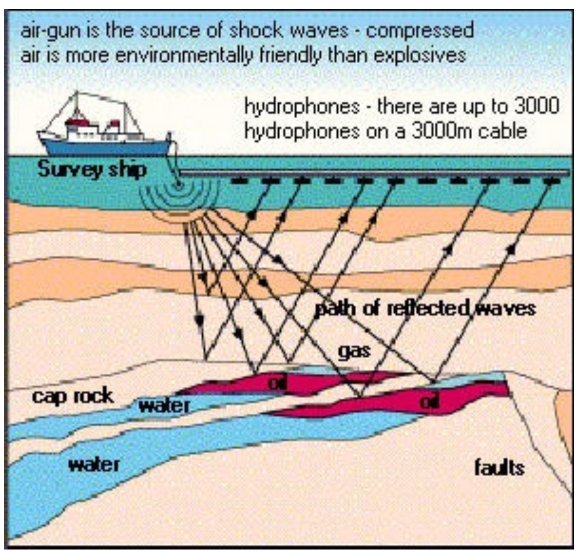
If there is a nonzero vector v and a real number λ such that

$$T(v) = \lambda v$$

then v is called an eigenvector for T and λ is called an eigenvalue for T.

Note: If v is an eigenvector then all nonzero multiples of v are eigenvectors $\rightsquigarrow$ line of eigenvectors

Applications



and many,
many more

Examples

1. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (-x,y)$ reflection over y-axis.

eigenvectors: $\{(0,y) : y \neq 0\}$ and $\{(x,0) : x \neq 0\}$
eigenvalues: $+1$ and -1.

2. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (2x,2y)$ scale by 2.

eigenvectors: $\mathbb{R}^2 - \{0\}$
eigenvalues: 2

or: $T(x,y) = (3x,2y)$

EXAMPLES

3. $T:\mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ rotation by θ

eigenvectors: $\mathbb{R}^2 - \{0\}$ if $\theta = 0, \pi$, $\emptyset$ otherwise
eigenvalues: 1 or nothing.

4. $T:\mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x,0)$ projection to x-axis.

eigenvectors: $\{(x,0) : x \neq 0\}$
eigenvalues: 1

EXAMPLES

5. $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (-x,-y,z)$ rotation by π about z-axis

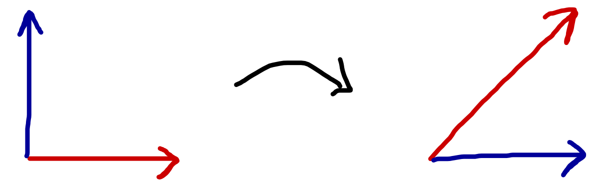
eigenvectors: $\{(x,y) \neq (0,0)\}$, $\{(0,0,z) : z \neq 0\}$
eigenvalues: -1 and 1

6. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (y,x)$ flip over $y=x$

eigenvectors: $\{(x,x) : x \neq 0\}$, $\{(x,x) : x \neq 0\}$
eigenvalues: 1 and -1

A More Complicated Example

7. $T(x,y) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x+y, x)$



What are the eigenvectors? Find them algebraically. Solve:

$$T(v) = \lambda v$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

There is a nonzero solution if and only if

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

or:

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

A More Complicated Example

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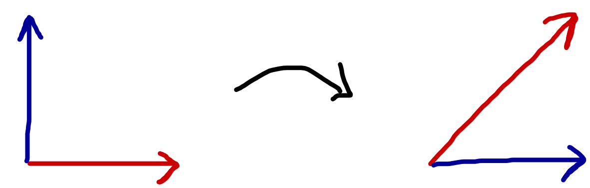


By the above calculation, the only possible eigenvalues are $\lambda = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

Are there any eigenvectors with these eigenvalues? Solve:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightsquigarrow \begin{aligned} x + y &= \lambda x \\ x &= \lambda y \end{aligned} \rightsquigarrow \begin{pmatrix} \lambda y \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} (1+\sqrt{5})/2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} (1-\sqrt{5})/2 \\ 1 \end{pmatrix}$$

So we have eigenvectors of $\left(\frac{1+\sqrt{5}}{2}, 1\right)$ and $\left(\frac{1-\sqrt{5}}{2}, 1\right)$ with eigenvalues $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively.

Recipe for Finding Eigenvalues & Eigenvectors

Say A is an $n \times n$ matrix.

1. To find eigenvalues, solve

$$\det(A - \lambda I) = 0$$

Note: In general, eigenvalues are complex numbers.

2. For each eigenvalue λ solve

$$Av = \lambda v \quad \text{or} \quad (A - \lambda I)v = 0$$

The polynomial $\det(A - \lambda I)$ is called the characteristic polynomial of A. Its roots are exactly the eigenvalues of A.

Finding Eigenvalues & Eigenvectors

Find the eigenvalues and eigenvectors of the following matrices.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 10 & 6 \\ 6 & 4 \end{pmatrix} \quad \begin{pmatrix} 6 & 3 \\ -2 & -1 \end{pmatrix}$$

Solving the Characteristic Polynomial

Sometimes it is difficult to find the roots of the characteristic polynomial.

It sometimes works to guess roots. One strategy is to guess the divisors of the constant term (plus or minus).

Example. Find the eigenvalues and eigenvectors of:

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

Characteristic polynomial: $\lambda^3 - 6\lambda^2 + 11\lambda - 6$.
Guess: $\pm 1, \pm 2, \pm 3, \pm 6$ as roots...

Examples

$$\begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$

eigenvalues: 2, 4, 5

$$\begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

eigenvalues: 2, 3, 5

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

eigenvalues: 2, −1, −1

GOOGLE PAGE RANK

Say the internet has pages $P_1, \ldots, P_N$.

Denote the importance of the i^{th} page P_i by $I(P_i)$.

To determine importance, each page gets 1 vote, split equally amongst outgoing links, BUT votes from important pages get more weight. (chicken and egg??):

$$I(P_i) = \sum_j I(P_j)/l_j \quad \text{(*)}$$

l_j = #links from P_j

How to compute I? Make a matrix:

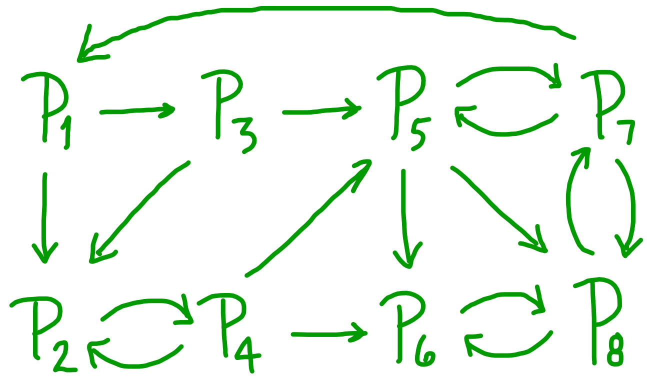
$$H_{ij} = \begin{cases} 1/l_j & P_j \text{ links to } P_i \\ 0 & \text{otherwise.} \end{cases}$$

Condition (*) is equivalent to: $H(I) = I$.

⇝ the importance vector I is an eigenvector for H with eigenvalue 1.

Google Page Rank

Example.



$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 1 & 1/3 & 0 \end{pmatrix} \rightsquigarrow I = \begin{pmatrix} .06 \\ .07 \\ .03 \\ .06 \\ .09 \\ .20 \\ .18 \\ .31 \end{pmatrix}$$

$\rightsquigarrow$ So page 8 is most important.

Google Page Rank

In real life $N = 25$ billion!

How to find I?

Idea: Iterate. Take any vector v, say $v = e_1$.
The sequence $H^k(v)$ approaches (the line through) I.

In above example, $H^{60}(e_1) \sim I$.

Principle: A linear transformation pulls most vectors towards the (leading) eigenvector. See the next lecture!