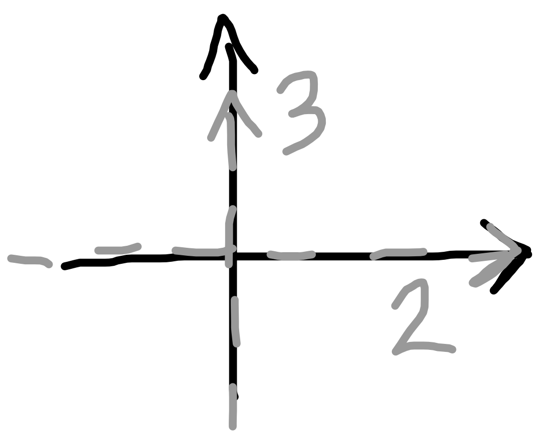
# 7.2 Diagonalization

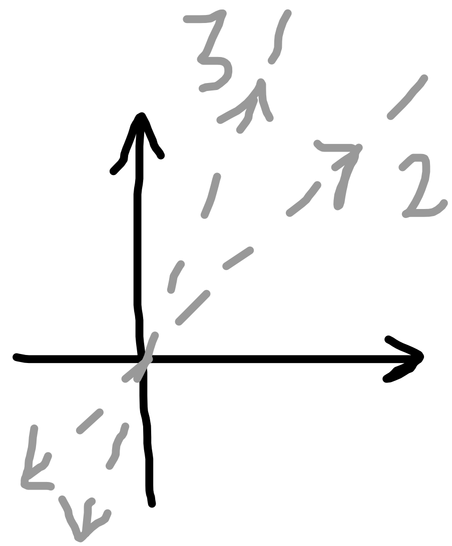
# Diagonalizing Matrices

What does $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ <u>do</u> to $\mathbb{R}^2$?

We find eigenvectors: $(2,1)$ and $(1,1)$
eigenvalues: $2$ $3$

It is similar to $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$



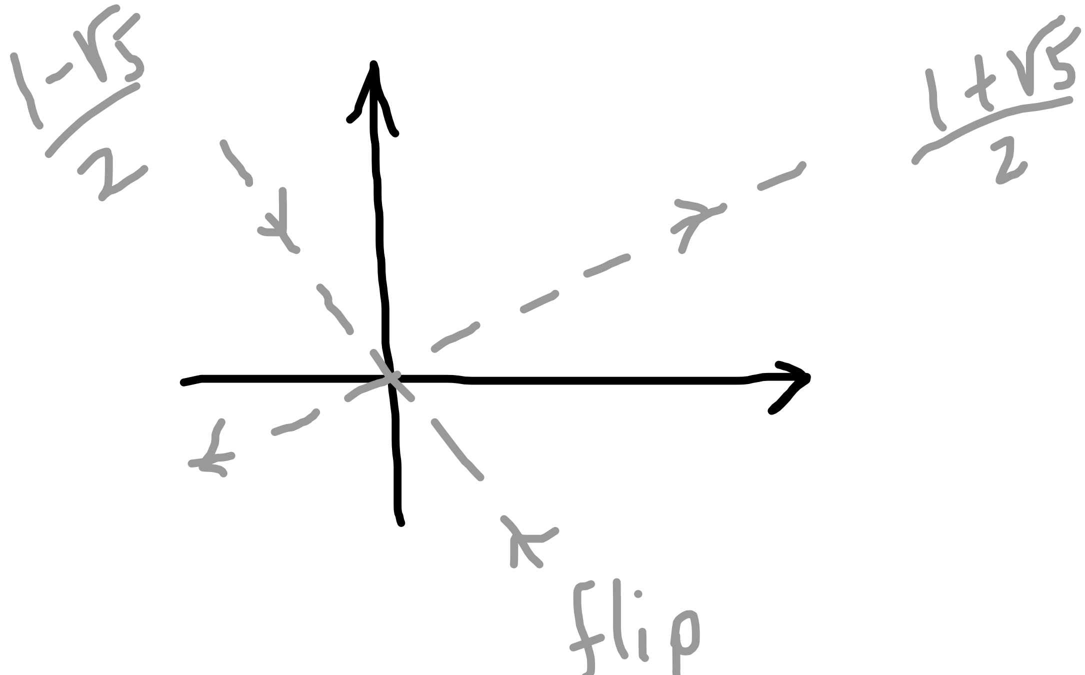




Similar means: doing the same thing, but with respect to different bases.

# Diagonalizing Matrices

What about $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$?

similar to $\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$

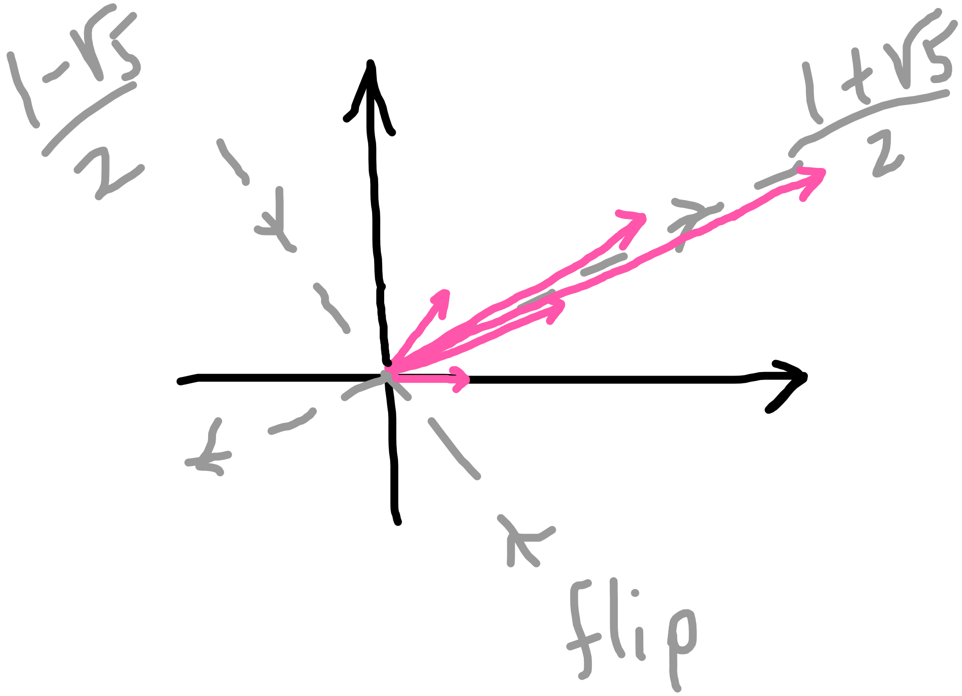



Similar means: doing the same thing, but with respect to different bases.

# Diagonalizing Matrices

What about powers of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$?

similar to $\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^k \end{pmatrix}$




$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

etc.

We conclude:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

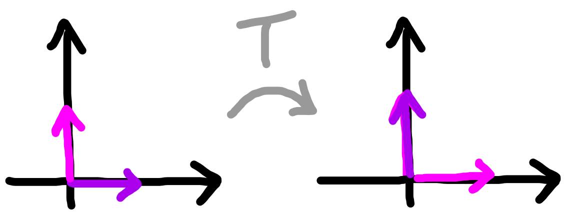
# Similar Matrices

Two matrices $A$ and $B$ are similar if there is a matrix $C$ so that

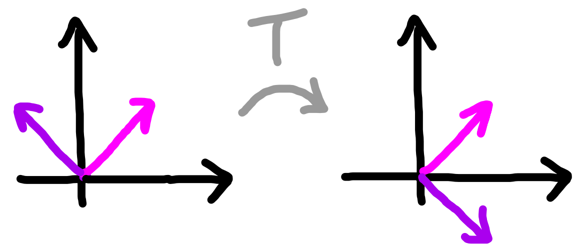
$$A = CBC^{-1}$$

This means that $A$ and $B$ are essentially the same, just written with respect to different bases.

EXAMPLE. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $y = x$. We write $T$ with respect to two different bases:




$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$




$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the change of basis

# Similar Matrices

Show that the following matrices are similar:

1. $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$

Hint: Write the preferred basis for one in terms of the preferred basis for the other, as in the previous example.

Use $C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$.

# Diagonalizable Matrices

A matrix is diagonalizable if it is similar to a diagonal matrix.

If a matrix $A$ is diagonalizable, it is easy to compute powers of $A$:

$$A = CDC^{-1}$$
$$\longrightarrow A^k = (CDC^{-1})^k$$
$$= CD^kC^{-1}$$

Computing $D^k$ is a snap:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{pmatrix}$$

So finding $A^{1000}$ only requires <u>two</u> matrix multiplications.

# Diagonalizable Matrices

1. Compute $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^5$.

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^5 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^5\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32 & 0 \\ 0 & 243 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -179 & 422 \\ -211 & 454 \end{pmatrix}$$

2. We saw $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$. Use this to find an explicit formula for $F_n$. How does this relate to our old method?

# Eigenvalues and Similarity

**Theorem.** Similar matrices have the same eigenvalues.

**Proof.** Say $B = CAC^{-1}$.

$$\det(B-\lambda I) = \det(CAC^{-1}-\lambda I) = \det(CAC^{-1} - \lambda CIC^{-1})$$
$$= \det(C(A-\lambda I)C^{-1}) = \det(C)\det(A-\lambda I)\det(C^{-1})$$
$$= \det(A-\lambda I).$$

**Theorem.** If a matrix $A$ is similar to a diagonal matrix $D$, the eigenvalues of $A$ are the same as the diagonal entries of $D$.

# Diagonalizable?

How do we know if a matrix $A$ is diagonalizable?

The algebraic multiplicity of an eigenvalue $\lambda$ for $A$ is the number of times $\lambda$ appears as a root of the characteristic polynomial $\det(A-\lambda I)$.

Example. The algebraic multiplicity of 5 in $(\lambda-5)^2(\lambda-1)$ is 2.

The geometric multiplicity of an eigenvalue $\lambda$ for $A$ is the number of free parameters in the solution of $(A-\lambda I)v = 0$.

This is the dimension of the eigenspace for $\lambda$.

THEOREM. A square matrix is diagonalizable if and only if each eigenvalue's algebraic and geometric multiplicities are equal.

# DIAGONALIZABLE?

THEOREM. A square matrix is diagonalizable if and only if each eigenvalue's algebraic and geometric multiplicities are equal.

Two restatements:

THEOREM. An $n \times n$ matrix is diagonalizable if and only if it has $n$ linearly independent eigenvectors.

THEOREM. A matrix is diagonalizable if and only if each eigenvalue of multiplicity $k$ has $k$ linearly independent eigenvectors.

# DIAGONALIZABLE?

THEOREM. A square matrix is diagonalizable if and only if each eigenvalue's algebraic and geometric multiplicities are equal.

We have:

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}$$

Therefore, if the algebraic multiplicity of $\lambda$ is 1, there is nothing to check.

If all algebraic multiplicities are 1, there is <u>really</u> nothing to check:

COROLLARY. If an $n \times n$ matrix has $n$ distinct eigenvalues, it is diagonalizable.

# Diagonalizable?

1. Is $\begin{pmatrix} 2 & -3/2 \\ 0 & 1/2 \end{pmatrix}$ diagonalizable?

   Yes. Eigenvectors are $(1,0)$ and $(1,1)$.

2. Is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable?

   No. All eigenvectors on x-axis.

3. Is $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ diagonalizable?

   Yes. Two distinct eigenvalues.

# Diagonalization Recipe

Say $A$ is diagonalizable, so $A = CDC^{-1}$. How to find $C$ and $D$?

- Put the eigenvalues of $A$ in some order: $\lambda_1, \dots, \lambda_n$.
- Choose $n$ linearly independent eigenvectors $v_1, \dots, v_n$, where the eigenvalue for $v_i$ is $\lambda_i$.

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \qquad C = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

Then need to find $C^{-1}$.

Why this works: $AC = CD$

$j$th col of $AC = A \cdot j$th col of $C$

$j$th col of $CD = \lambda_j \cdot j$th col of $C$

# Diagonalization Recipe

Diagonalize the following matrices:

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$$

Recall: To find $C^{-1}$, write

$$(C \mid I)$$

Row reduce:

$$(I \mid C^{-1})$$

# Diagonalization

Are the following matrices diagonalizable? If so, diagonalize.

$$\begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

No.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

No.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Yes.