Linear Programming

Linear Programs

Example. Maximize $3x+2y = z$ ← objective function
subject to $2x + y \leq 20$
$x, y \geq 0$ ← constraints

e.g. x = widget, y = gadget — 2 hrs to make widget
3 = profit on widget — 1 hr to make gadget
2 = profit on gadget — 20 hours available time

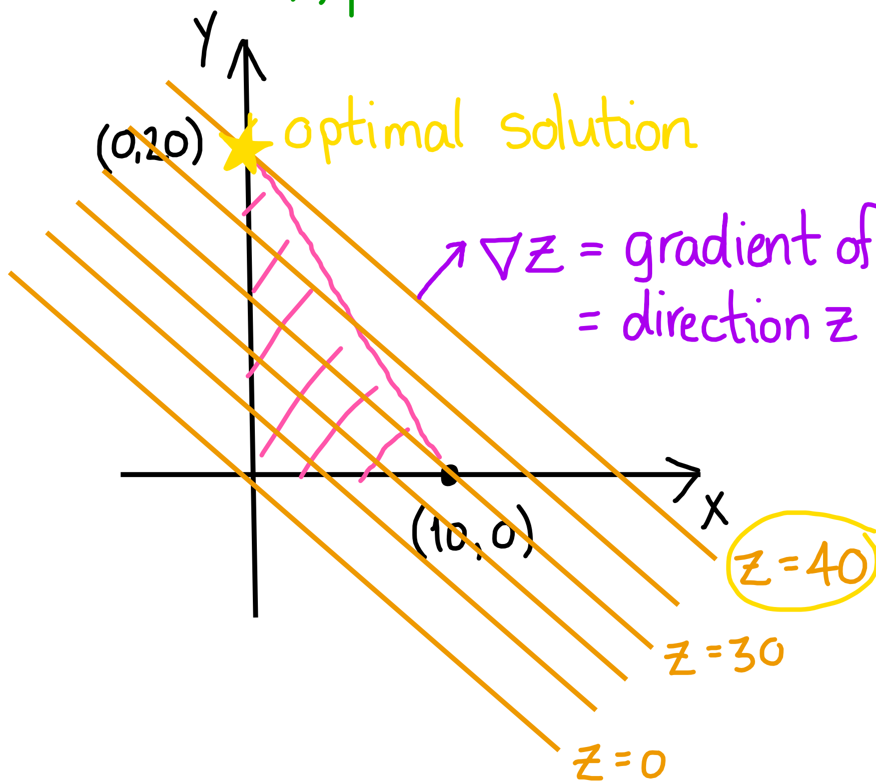
A maximization (or mimimization) problem where the objective function and constraints are all linear is called a linear programming problem.

How to solve?

$3x+2y$ is linear
$(x+y)^3$, $3\ln(x)$, $\sqrt{x+y}$
are not linear

Linear Programs

Example. Maximize $3x+2y = z$ ← objective function
subject to $2x + y \leq 20$, $x, y \geq 0$ ← constraints



Pink triangle = feasible region

LINEAR PROGRAMS

EXAMPLE. Maximize $z = x + y$

subject to $2x + 3y \le 6$

$4x + 2y \le 8$

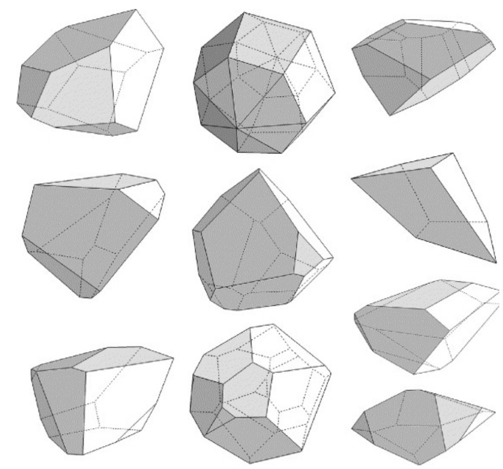
$x, y \ge 0$

Answer: $z = 6.5$

Notice: The optimum always occurs at a corner.

This always works! To find the optimum, we move the objective function hyperplane in the direction of its perpendicular (gradient) and observe the last point(s) of the feasible region it passes through. This will always be at a corner.

The Feasible Region



The feasible region for a linear program is the intersection of finitely many half-spaces. Thus, it is a convex (possibly infinite) polyhedron.

We deduce:

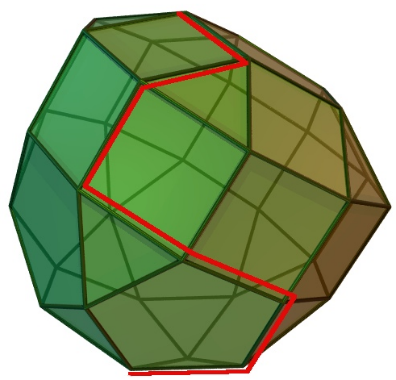
Fact. If a finite optimum exists for a linear program, then there is an optimal extreme point (= corner).

In other words, to find optima, it is enough to look at corners. But simply checking all corners takes way too long!

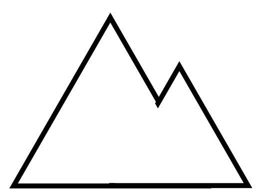
Example. Maximize $z = 3x$ s.t. $0 \leq x, y \leq 1$.

The Simplex Method

The basic idea: Start at some corner of the feasible region. See if any adjacent corners are higher (in z-value). If so, move to that corner. If not, stop.



In other words, if you always move up, you eventually get to the top. This does not work for nonconvex shapes:



Now: How to formalize this?

The Simplex Method

The simplex method was devised in 1947 by George Dantzig, of the RAND corporation.

George Dantzig

It was deemed one of the top ten algorithms of the 20th century in the Jan/Feb 2000 issue of Computing in Science and Engineering.

Standard Form

Given a linear program, we put it in standard form by adding slack (or surplus) variables so that all inequalities become equalities.

Example. The standard form of

$$\begin{array}{ll} \text{maximize} & z = x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 \le 4 \\ & x_1 + 2x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{array}$$

is:

$$\begin{array}{ll} \text{maximize} & z = x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 + x_3 = 4 \\ & x_1 + 2x_2 + x_4 = 3 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

Standard Form

The standard form: maximize $z = x_1 + x_2$

subject to
$$\begin{aligned} 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

gives a system of linear equations:
$$\begin{aligned} z - x_1 - x_2 \qquad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

subject to the condition $x_i \geq 0$.

So we are looking at the intersection of an $(n-k)$-plane with the positive orthant of $\mathbb{R}^n$, assuming there are $n-1$ of the x_i and $k-1$ original constraints (not counting positivity).

We usually draw the projection that kills the slack variables.

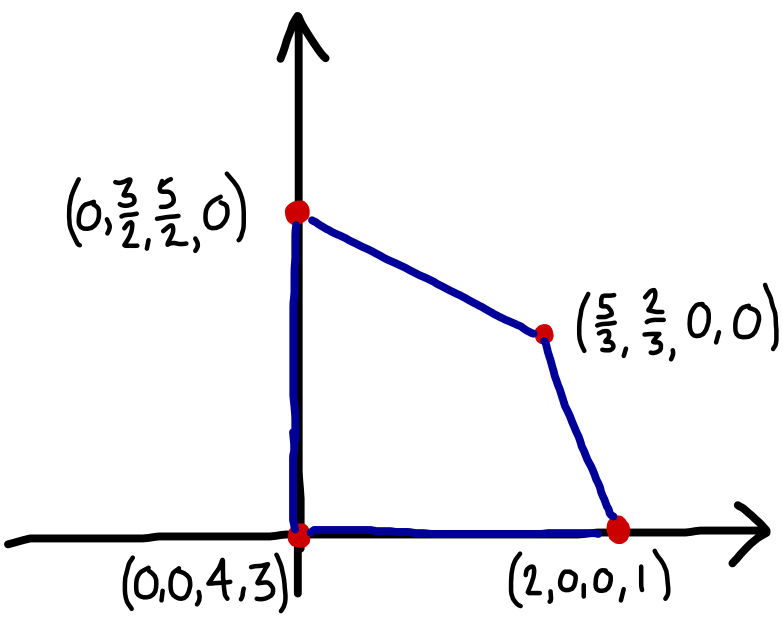
Standard Form

The picture of the feasible region for

$$\begin{aligned} z - x_1 - x_2 \qquad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

$x_i \geq 0.$

is:



The Simplex Method

After putting in standard form, we want to maximize z, where:

$$\begin{aligned} z - x_1 - x_2 \qquad\qquad &= 0 \\ 2x_1 + x_2 + x_3 \qquad &= 4 \\ x_1 + 2x_2 \qquad + x_4 &= 3 \end{aligned}$$

and $x_i \geq 0$.

In these notes, we assume no surplus variables (i.e. all original constraints are $\leq$) and all right-hand sides are positive.

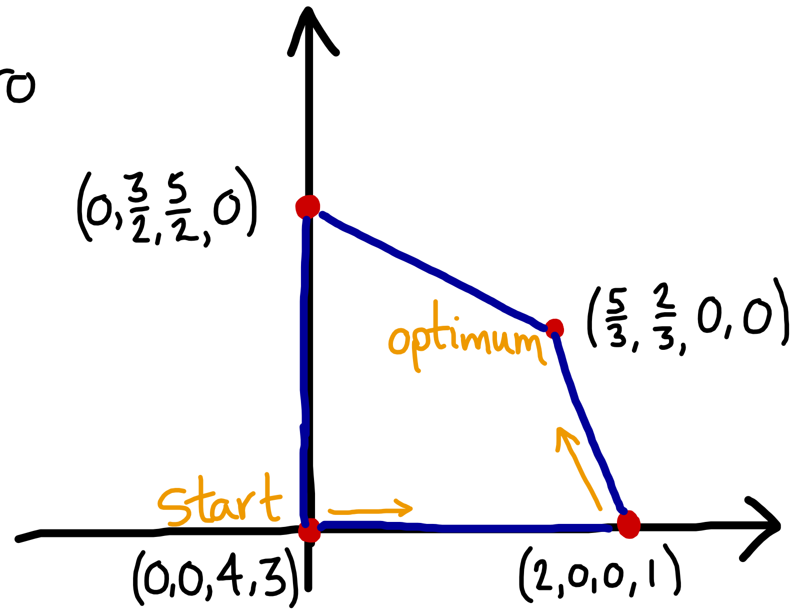
A basic variable is one that appears in only one equation.

Rule 0. Setting all nonbasic variables equal to 0 gives a corner of the feasible region, called a basic solution.

In the above example, the basic solution is $z = 0$ at $(0,0,4,3)$.

The Simplex Method

As we can see, by changing one nonzero coordinate to be zero, we move along an edge of the feasible region. So to make progress from our starting point to the optimal solution, we change the basic variables, one at a time.



We just need to make sure ① We are always increasing z
and ② We stay in the feasible region

Rules ① and ② below address these two points.

The Simplex Method

Say we have a linear program in standard form:

$$\begin{aligned} z - x_1 - x_2 \qquad\quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \qquad + x_4 &= 3 \end{aligned}$$

Rule 1. The current basic solution is optimal if and only if all variables in the top row have nonnegative coefficients.

If the current basic solution is not optimal, we choose a variable with negative coefficient in the top row and make it basic using row operations.

But, we need to do this carefully!

In Rule 1, we usually choose a variable with <u>most</u> negative coeff.

THE SIMPLEX METHOD

Say we have a linear program in standard form:

$$
\begin{aligned}
z - x_1 - x_2 \phantom{{}+x_3 +x_4} &= 0 \\
2x_1 + x_2 + x_3 \phantom{{}+x_4} &= 4 \\
x_1 + 2x_2 \phantom{{}+x_3} + x_4 &= 3
\end{aligned}
$$

Say, by Rule 1, we decide to make x_1 basic. This means we want to use row operations to remove x_1 from all equations but one. Which to choose?

RULE 2. When making x_i basic, we leave x_i in the row where

$$\frac{\text{RHS}}{\text{coeff}(x_1)}$$

is the smallest positive number among all rows.

positive $\rightsquigarrow$ z will increase   smallest $\rightsquigarrow$ stay in feasible region

The Simplex Method

Rules 1 and 2 comprise the simplex method. Let's apply it to our example.

$$
\begin{aligned}
z - x_1 - x_2 \qquad\qquad &= 0 \\
2x_1 + x_2 + x_3 \qquad &= 4 \\
x_1 + 2x_2 \qquad + x_4 &= 3
\end{aligned}
$$

By Rule 1, we are not at the optimum, and we choose to make x_1 basic.

By Rule 2, we use the $2x_1$ as our pivot, and get:

$$
\begin{aligned}
z \quad - \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 \qquad &= 2 \\
x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 \qquad &= 2 \\
\tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 &= 1
\end{aligned}
$$

What happens if you use the bottom row as a pivot?

$\longrightarrow$ new basic solution $(2,0,0,1)$, $z=2$.

The Simplex Method

We now have:

$$
\begin{aligned}
z \quad -\tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 \quad &= 2 \\
x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 \quad &= 2 \\
\tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 &= 1
\end{aligned}
$$

By Rules 1 and 2, we make x_2 basic by pivoting from the bottom row. We get:

$$
\begin{aligned}
z \quad + \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 &= 7/3 \\
x_1 \quad + \tfrac{2}{3}x_3 - \tfrac{1}{3}x_4 &= 5/3 \\
x_2 - \tfrac{1}{3}x_3 + \tfrac{2}{3}x_4 &= 2/3
\end{aligned}
$$

$\rightsquigarrow$ basic solution $(5/3, 2/3, 0, 0)$ $z = 7/3$.

This is optimal by Rule 1!

TABLEAUX

We can succinctly record (and perform) the above calculation as follows:

z	x_1	x_2	x_3	x_4	RHS	Basic Soln
1	-1	-1	0	0	0	
0	2	1	1	0	4	$(0,0,4,3)$ $z=0$
0	1	2	0	1	3	
1	0	$-1/2$	$1/2$	0	2	
0	1	$1/2$	$1/2$	0	2	$(2,0,0,1)$ $z=2$
0	0	$3/2$	$-1/2$	1	1	
1	0	0	$1/3$	$1/3$	$7/3$	
0	1	0	$2/3$	$-1/3$	$5/3$	$(5/3, 2/3, 0, 0)$ $z=7/3$
0	0	1	$-1/3$	$2/3$	$2/3$	

Note the basic solution is easily read from the RHS.

The Simplex Method

Problem. Maximize $4x_1 + x_2 - x_3 = z$

subject to

$$
\begin{aligned}
x_1 + \ 3x_3 &\le 6 \\
3x_1 + x_2 + 3x_3 &\le 9 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}
$$

First we write the standard form:

$$
\begin{aligned}
z - 4x_1 - x_2 + x_3 &= 0 \\
x_1 + 3x_3 + x_4 &= 0 \\
3x_1 + x_2 + 3x_3 + x_5 &= 9
\end{aligned}
$$

The Simplex Method

Problem. Maximize $4x_1 + x_2 - x_3 = z$

subject to
$$x_1 + 3x_3 \le 6$$
$$3x_1 + x_2 + 3x_3 \le 9$$
$$x_1, x_2, x_3 \ge 0$$

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic Soln
1	-4	-1	1	0	0	0	
0	1	0	3	1	0	6	(0,0,0,6,9) $z=0$
0	3	1	3	0	1	9	
1	0	1/3	5	0	4/3	12	
0	0	-1/3	2	1	-1/3	3	(3,0,0,3,0) $z=12$
0	1	1/3	1	0	1/3	3	

The Simplex Method

Problem. Maximize $z = x_1 + \frac{1}{2}x_2$

subject to

$$2x_1 + x_2 \le 4$$
$$x_1 + 2x_2 \le 3$$
$$x_1, x_2 \ge 0$$

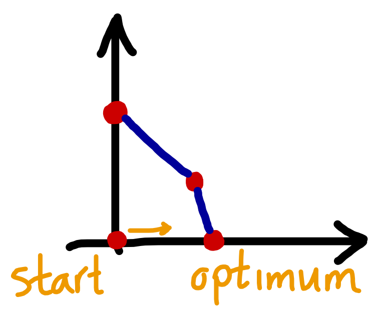
First we write the standard form:

$$\begin{aligned} z - x_1 - \tfrac{1}{2}x_2 \qquad\quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

The Simplex Method

$$z - x_1 - \tfrac{1}{2}x_2 = 0$$
$$2x_1 + x_2 + x_3 = 4$$
$$x_1 + 2x_2 + x_4 = 3$$

z	x_1	x_2	x_3	x_4	RHS	Basic Soln
1	−1	−1/2	0	0	0	
0	2	1	1	0	4	(0,0,4,3) $z=0$
0	1	2	0	1	3	
1	0	0	1/2	0	2	
0	1	1/2	1/2	0	2	(2,0,0,1) $z=2$
0	0	3/2	−1/2	1	1	



Note: If you pivot on x_2 instead of x_1, it takes 3 steps instead of 1!

The Simplex Method

Example. Maximize $z = 3x + 4y$
subject to $2x + y \le 5$
$x, y \geq 0$

$z = 20$ at $(0, 5)$

Example. Maximize $z = 3x + 2y$
subject to $x + y \le 4$
$2x + y \le 5$
$x, y \geq 0$

$z = 9$ at $(1, 3)$

Example. Maximize $z = 3x + 2y$
subject to $2x + y \le 18$
$2x + 3y \le 42$
$3x + 2y \le 24$
$x, y \geq 0$

$z = 24$ at $(8, 0)$

Example. Maximize $z = x_1 + 2x_2 - x_3$
subject to $2x_1 + x_2 + x_3 \le 14$
$4x_1 + 2x_2 + 3x_3 \le 28$
$2x_1 + 5x_2 + 5x_3 \le 30$
$x_i \geq 0$

$z = 13$ at $(5, 4, 0)$

The Simplex Method

Tableau for last example:

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Solution
1	-1	-2	1	0	0	0	0	
0	2	1	1	1	0	0	14	$(0,0,0)$ $z=0$
0	4	2	3	0	1	0	28	
0	2	5	5	0	0	1	30	
1	$-1/5$	0	3	0	0	$2/5$	12	
0	$8/5$	0	0	1	0	$-1/5$	8	$(0,6,0)$ $z=12$
0	$16/5$	0	1	0	1	$-2/5$	16	
0	$2/5$	1	1	0	0	$1/5$	6	
1	0	0	$49/16$	0	$1/16$	$3/8$	13	
0	0	0	$-1/2$	1	$-1/2$	0	0	$(5,4,0)$ $z=13$
0	1	0	$5/16$	0	$5/16$	$-1/8$	5	
0	0	1	$7/8$	0	$-1/8$	$1/4$	4	

The Simplex Method

Example. A company produces chairs and sofas.
A chair requires 3 hrs carpentry, 9 hrs finishing, 2 hrs upholstery.
A sofa requires 2 hrs carpentry, 4 hrs finishing, 10 hrs upholstery.
The company can afford 66 hours of carpentry, 180 hours of finishing, and 200 hours of upholstery.
The profit on a chair is \$90 and on a sofa is \$75.
How many chairs and sofas should be made to maximize profit?

Solution. Want to maximize $P = 90x_1 + 75x_2$

subject to

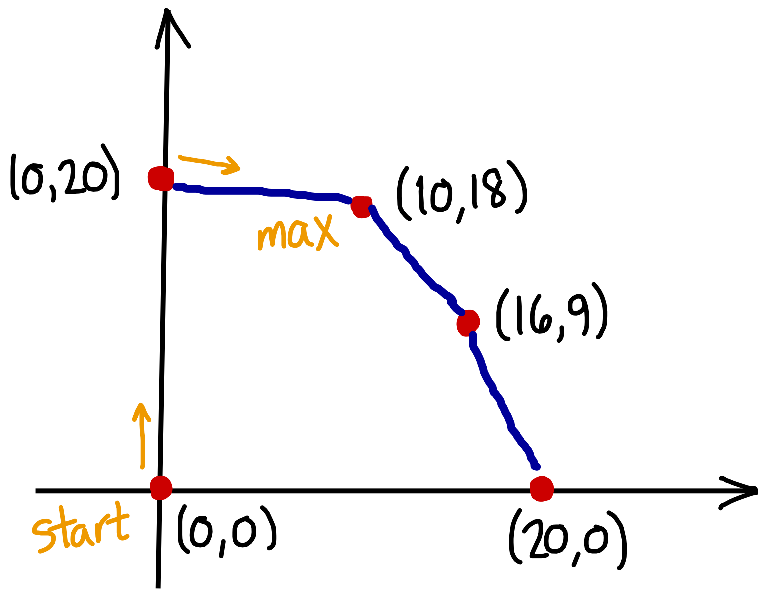
$$3x_1 + 2x_2 \leq 66$$
$$9x_1 + 4x_2 \leq 180$$
$$2x_1 + 10x_2 \leq 200$$

The Simplex Method

Z	X_1	X_2	X_3	X_4	X_5	RHS	Solution
1	-90	-75	0	0	0	0	
0	3	2	1	0	0	66	(0,0) P=0
0	9	4	0	1	0	180	
0	2	10	0	0	1	200	
1	-75	0	0	0	15/2	1500	
0	13/5	0	1	0	-1/5	26	(0,20) P=1500
0	41/5	0	0	1	-2/5	100	
0	1/5	1	0	0	1/10	20	
1	0	0	375/13	0	45/26	2250	
0	1	0	5/13	0	-1/13	10	(10,18) P=2250
0	0	0	-41/13	1	41/65	18	
0	0	1	-1/13	0	15/130	18	

The Simplex Method

Picture for the last problem:



Geometry of the Simplex Method

We can write the constraints of a linear program as $Ax = b$.

The vertices of the feasible region are points x that lie in the feasible region and where the columns of A corresponding to the nonzero entries of x are linearly independent.

Two vertices span an edge if they are nonzero in all of the same coordinates except for 1.

Thus, swapping one basic variable for another corresponds to moving along an edge.

As we perform row operations, the feasible region (and objective function) changes, but the corresponding graph is naturally isomorphic to the original.