SECTION 5.3

Solving Recurrence Relations: The Characteristic Polynomial

WHY STUDY RECURRENCE RELATIONS?

Reason #1: Sometimes a sequence of numbers is more easily described this way, e.g.: the number of moves in our solution to the Towers of Hanoi problem is $a_n = 2a_{n-1} + 1$

Also, the number of Fibonacci rabbits: $a_n = a_{n-1} + a_{n-2}$

Reason #2: They are discrete versions of differential equations:

$$a'_n = a_n - a_{n-1} \qquad a''_n = a'_n - a'_{n-1}$$

So differential equations can be approximated by a difference equation, then converted to a recurrence relation.

SOLVING RECURRENCE RELATIONS

To <u>solve</u> a recurrence relation means to give an explicit formula.

Example: $a_n = a_{n-1} + 2, \quad a_0 = 1$

Solution: $a_n = 2n + 1$

Can use induction to <u>prove</u> this is a solution:

Base case: $a_0 = 1 = 2 \cdot 0 + 1$

Assume: $a_k = 2k + 1$

Show $a_{k+1} = 2(k+1) + 1$:

$$\begin{aligned} a_{k+1} &= a_k + 2 \\ &= (2k+1) + 2 \\ &= 2(k+1) + 1 \quad \checkmark \end{aligned}$$

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

$$a_n = r a_{n-1} + s a_{n-2}$$

Second order: a_n defined in terms of a_{n-1}, a_{n-2}

Linear: A <u>linear</u> combination of x and y is

$$5x - 2y$$

not

$$5xy \quad \text{or} \quad e^x \quad \text{or} \quad \sqrt{x+y}$$

Homogeneous: No "extra stuff" after the linear combination of a_{n-1} and a_{n-2}.

Extra stuff = function of n.

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} + a_{n-2}, \quad a_0 = 0, a_1 = 1$

What is the solution?

First few terms: $0, 1, 2, 5, 12, 29, 70, 169, \ldots$

What is the pattern?

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

It turns out we can solve them all!

Theorem: Consider the recurrence relation

$$a_n = r a_{n-1} + s a_{n-2}.$$

Let b_1, b_2 be the roots of

$$x^2 - rx - s$$

Then the solution to a_n is:

$$a_n = \begin{cases} c_1 b_1^n + c_2 b_2^n & \text{if } b_1 \neq b_2 \\ c_1 b_1^n + c_2 n b_2^n & \text{if } b_1 = b_2 \end{cases}$$

The c_i are determined by the initial conditions.

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = a_{n-2}$, $a_0 = 1$, $a_1 = 3$.

We can write this as: $a_n = 0 \cdot a_{n-1} + a_{n-2}$

$$\rightsquigarrow x^2 - 0 \cdot x - 1 = x^2 - 1 = (x+1)(x-1)$$

So $b_1 = 1$, $b_2 = -1$

By the theorem:

$$a_n = c_1(1)^n + c_2(-1)^n$$
$$= c_1 + c_2(-1)^n$$

Find c_1, c_2 using initial conditions:

$$a_0 = 1 = c_1 + c_2$$
$$a_1 = 3 = c_1 - c_2$$

$\rightsquigarrow c_1 = 2$, $c_2 = -1$

$\rightsquigarrow a_n = 2 + (-1)(-1)^n = 2 + (-1)^{n+1}$

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1, a_1 = 0$

$\rightsquigarrow x^2 - 6x + 9 \rightsquigarrow (x-3)^2 \rightsquigarrow b_1 = b_2 = 3$

$\rightsquigarrow a_n = c_1 3^n + c_2 n 3^n$

Use the initial conditions to find the c_i:

$$a_0 = c_1 = 1$$
$$a_1 = 3c_1 + 3c_2 = 3 + 3c_2 = 3(1 + c_2) = 0$$

$\rightsquigarrow c_1 = 1,\ c_2 = -1$

So: $a_n = 3^n - n3^n$

THE CASE $b_1 = b_2$

$$b_1 = b_2$$

$$\iff (x-b_1)^2 = x^2 - 2b_1 x + b_1^2$$

$$\iff a_n = 2b_1 a_{n-1} - b_1^2 a_{n-2}$$

$$\iff a_n = r a_{n-1} + s a_{n-2}$$

where $s = -r^2/4$

MORE PROBLEMS

① Solve $a_n = 9a_{n-2}$ where

(a) $a_0 = 6,\ a_1 = 12$

(b) $a_0 = 6,\ a_2 = 54$

(c) $a_0 = 6,\ a_2 = 10$

② Solve $a_n = 8a_{n-1} - 16a_{n-2},\ a_0 = 1,\ a_1 = 16$

③ Solve $5a_n = 11a_{n-1} - 2a_{n-2},\ a_0 = 2,\ a_1 = -8.$

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

General form: $a_n = r a_{n-1} + s a_{n-2} + f(n)$

Examples:

$$a_n = 2a_{n-1} + 1$$
$$a_n = 3a_{n-1} + 2a_{n-2} + n$$
$$a_n = 5a_{n-1} - a_{n-2} + 2^n$$
$$a_n = a_{n-1} + a_{n-2} + (n^7 + n^n + n!)$$

We do not know how to solve them all, but...

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

THEOREM: Let $a_n = ra_{n-1} + sa_{n-2} + f(n)$.
Let p_n be any particular solution to a_n.
Let q_n be the general solution to $q_n = rq_{n-1} + sq_{n-2}$.
Then $p_n + q_n$ is the general solution to a_n.

We already have a sure-fire way to find q_n.

The hard part is that we don't know how to find p_n — we have to <u>guess</u>.

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

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Then $p_n + q_n$ is the general solution to a_n.

Proof that $p_n + q_n$ really is a solution:

By definition: $p_n = rp_{n-1} + sp_{n-2} + f(n)$
$q_n = rq_{n-1} + sq_{n-2}$

Let $t_n = p_n + q_n$. Adding the last two lines:
$t_n = rt_{n-1} + st_{n-2} + f(n)$ ✓

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = 2a_{n-1} + 1$

First we solve $q_n = 2q_{n-1}$

$\longrightarrow x^2 - 2x \longrightarrow x = 0, 2$

$\longrightarrow x = k2^n$

Then we find a particular solution to a_n by "guessing":

$a_n = -1$

Check: $-1 = 2 \cdot (-1) + 1$ ✓

By the theorem, the general solution is:

$a_n = k2^n - 1$

We find k using initial conditions.

HOW TO GUESS PARTICULAR SOLUTIONS

If $f(n)$ is...	Guess p_n to be...
exponential	exponential (same base)
linear	linear
quadratic	quadratic
n^{th} degree polynomial	n^{th} degree polynomial
anything else	???

Example: Solve $a_n = 3a_{n-1} + 5\cdot 7^n$, $a_0 = 2$.

First we "guess" p_n:

$$p_n = c7^n$$

Need to find c: $c7^n = 3c7^{n-1} + 5\cdot 7^n$

$$7^{n-1}(7c - 3c - 5\cdot 7) = 0$$

$$c = 35/4$$

$$\rightsquigarrow p_n = 35/4\, 7^n = 5/4\, 7^{n+1}$$

Then we solve $q_n = 3q_{n-1} \rightsquigarrow q_n = k3^n$

By the theorem $a_n = p_n + q_n = k3^n + 5/4\, 7^{n+1}$

Now we find k: $2 = a_0 = k + 35/4$

$$k = -27/4$$

$$\rightsquigarrow a_n = -27/4 \cdot 3^n + 5/4\, 7^{n+1} = -\frac{1}{4}3^{n+3} + \frac{5}{4}7^{n+1}$$

Example: $a_n = -a_{n-1} + n, \quad a_0 = 1/4.$

First we guess $p_n = mn + b$ Need to find m, b:

$$\begin{aligned} mn + b &= -(m(n-1)+b) + n \\ &= -(mn - m + b) + n \\ &= -mn + m - b + n \\ &= (1-m)n + (m-b) \end{aligned}$$

$\rightsquigarrow m = 1/2, \quad b = 1/4$

$\rightsquigarrow p_n = \frac{1}{2}n + \frac{1}{4}$

Then we solve $q_n = -q_{n-1} \rightsquigarrow q_n = k(-1)^n$

By the theorem: $a_n = k(-1)^n + (1/2\, n + 1/4)$

Using initial condition: $a_0 = 1/4 = k + 1/4 \rightsquigarrow k = 0$

So: $a_n = n/2 + 1/4.$

MORE PROBLEMS

① Solve $a_n = 5a_{n-1} - 6a_{n-2} + 6\cdot 4^n$

② Solve $a_n = a_{n-1} + 3n^2$, $a_0 = 7$

By the way, there is another method for solving #2, the method of undetermined coefficients. Idea: recursively substitute: $a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum i^2 = \cdots$