SECTION 5.4

SOLVING RECURRENCE RELATIONS–GENERATING FUNCTIONS

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} - n/3$ (actually, this is first order)

Steps:

1. Solve $q_n = 2q_{n-1}$ (general solution)
2. Find one particular solution p_n to $p_n = 2p_{n-1} + n/3$
   guess: $p_n = mn + b$
3. Add $p_n + q_n$
4. Solve for constants

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} - n/3, \quad a_0 = 1$

① $q_n = 2q_{n-1}$

$\rightsquigarrow q_n = c2^n$

② Guess: $p_n = mn + b$ Need to find m, b

$mn + b = 2(m(n-1)+b) - n/3$

$mn + b = 2mn - 2m + 2b - n/3$

$mn + b = (2m - \frac{1}{3})n + (2b - 2m)$

$\rightsquigarrow m = 2m - 1/3 \rightsquigarrow m = 1/3$

$b = 2b - 2m \rightsquigarrow b = 2m = 2/3$

So $p_n = n/3 + 2/3$

③ $a_n = p_n + q_n = c2^n + n/3 + 2/3$

④ $a_0 = c + 2/3$

$\rightsquigarrow c = 1/3$

$a_n = (2^n + n + 2)/3$

GENERATING FUNCTIONS

Sometimes counting problems, or recurrence relations can be solved using polynomials in a clever way.

Example: Find the number of solutions of

$$a+b+c=10$$

where a is allowed to be 2, 3, or 4
b is allowed to be 3, 4, or 5
c is allowed to be 1, 3, or 4

The answer is the coefficient of x^{10} in

$$(x^2+x^3+x^4)(x^3+x^4+x^5)(x+x^3+x^4)$$

e.g. $2+5+3 \longleftrightarrow x^2 x^5 x^3$

This problem can be solved with a computer algebra system.

GENERATING FUNCTIONS

The generating function for the sequence

$$a_0, a_1, a_2, a_3, \ldots$$

is

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

For example

$$a_n = 1 \longleftrightarrow 1, 1, 1, 1, \ldots \longleftrightarrow 1 + x + x^2 + x^3 + \cdots$$

$$a_n = n + 1 \longleftrightarrow 1, 2, 3, 4, \ldots \longleftrightarrow 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$a_n = n \longleftrightarrow 0, 1, 2, 3, \ldots \longleftrightarrow x + 2x^2 + 3x^3 + \cdots$$

POWER SERIES

A generating function, as an object, is what is called a power series, that is, a formal sum ← Think "string"

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

These can be added, subtracted, and multiplied:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$
$$f(x)g(x) = a_0 b_0 + (a_1 b_0 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

But we never plug in numbers for x, like with Taylor Series.

So generating functions should not be thought of as functions!

POWER SERIES

What about dividing?

Amazingly, yes! as long as $a_0 \neq 0$.

$\frac{1}{f(x)}$ is the generating function so that $f(x) \cdot \frac{1}{f(x)} = 1$

Example: $f(x) = 1 + x + x^2 + \cdots$

What is a power series that, when multiplied by $f(x)$ gives 1?

$(1-x)\,f(x) = 1 + 0x + 0x^2 + \cdots = 1 \rightsquigarrow \frac{1}{f(x)} = 1-x$, or $f(x) = \frac{1}{1-x}$

We say $1/1-x$ is the generating function for $a_n = 1$.

EXAMPLES OF GENERATING FUNCTIONS

$$\frac{1}{1-x} = 1+x+x^2+\cdots \quad \longleftrightarrow \quad a_n = 1$$

$$\frac{1}{1+x} = 1-x+x^2-\cdots \quad \longleftrightarrow \quad a_n = (-1)^n$$

$$\frac{1}{1-ax} = 1+bx+b^2x^2+\cdots \quad \longleftrightarrow \quad a_n = b^n$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots \quad \longleftrightarrow \quad a_n = n+1$$

* The other three follow from the first one by substitution & differentiation

What is the generating function for $a_n = n$?

$$a_n = n \Longleftrightarrow x+2x^2+3x^3+\cdots = \frac{x}{(1-x)^2}$$

What about $a_n = -2n$?

$$-2x/(1-x)^2$$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS

Example: $a_n = 2a_{n-1}, \; a_0 = 1$

The generating function for a_n is:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

Using $a_n = 2a_{n-1}$, and $a_0 = 1$, we can rewrite each term of $f(x)$:

$$a_0 = 1$$
$$a_1 x = 2a_0 x$$
$$a_2 x^2 = 2a_1 x^2$$
$$a_3 x^3 = 2a_2 x^3$$
$$\vdots$$

Add up:

$$f(x) = 1 + 2x\, f(x)$$

Solve for $f(x)$:

$$f(x) = 1/1-2x \longleftrightarrow a_n = 2^n$$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS

Example: $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

Start with $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

Then

$$\begin{aligned}
a_0 &= 2 \\
a_1 x &= -x \\
a_2 x^2 &= 2a_1 x^2 \qquad - a_0 x^2 \\
a_3 x^3 &= 2a_2 x^3 \qquad - a_1 x^3 \\
&\vdots
\end{aligned}$$

Add up: $f(x) = \left(2x\, f(x) + 2 - 5x\right) - x^2 f(x)$

$$\leadsto f(x) = \frac{2-5x}{(1-2x-x^2)} = \frac{2}{(1-x^2)^2} - \frac{5x}{(1-x^2)^2}$$

$$\leadsto a_n = 2(n+1) - 5n = -3n + 2.$$

PARTIAL FRACTIONS

Example: Rewrite $\frac{1-x}{1-5x+6x^2}$ as a sum of fractions where the denominator is linear.

$$\frac{1-x}{1-5x-6x^2} = \frac{1-x}{(1-3x)(1-2x)} = \frac{A}{1-3x} + \frac{B}{1-2x}$$

$$\rightsquigarrow A(1-2x) + B(1-3x) = 1-x$$

$$x = 1/2 \rightsquigarrow B\left(1-\tfrac{3}{2}\right) = \tfrac{1}{2} \rightsquigarrow B = -1$$

$$x = 1/3 \rightsquigarrow A(1-2/3) = 2/3 \rightsquigarrow A = 2$$

$$\frac{1-x}{1-5x+6x^2} = \frac{2}{1-3x} - \frac{1}{1-2x}$$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: Solve $a_n = 5a_{n-1} - 6a_{n-2}$ $\quad a_0 = 1, a_1 = 4$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$\rightsquigarrow$

$$a_0 = 1$$
$$a_1 x = 4x$$
$$a_2 x^2 = 5a_1 x^2 - 6a_0 x^2$$
$$a_3 x^3 = 5a_2 x^3 - 6a_1 x^3$$
$$\vdots$$

Add up: $f(x) = 5x f(x) - x + 1 - 6x^2 f(x)$

$\rightsquigarrow f(x) = \dfrac{1-x}{1-5x+6x^2} = \dfrac{2}{1-3x} - \dfrac{1}{1-2x} \rightsquigarrow a_n = 2 \cdot 3^n - 2^n$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: Solve $a_n = a_{n-1} + a_{n-2}$ $\quad a_0 = 0,\ a_1 = 1$

As above, get: $f(x) = \frac{x}{1-x-x^2}$

$a = \frac{1+\sqrt{5}}{2},\ b = \frac{1-\sqrt{5}}{2}$

Partial fractions: $1 - x - x^2 = (1-ax)(1-bx)$

Note: $ab = -1,\ a+b = 1$

$a - b = \sqrt{5}$

$$f(x) = \frac{1/\sqrt{5}}{1-ax} - \frac{1/\sqrt{5}}{1-bx}$$

So $a_n = \frac{1}{\sqrt{5}}(a^n - b^n)$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: $a_n = 2a_{n-1} - n/3, \quad a_0 = 1$

Example: $a_n = a_{n-1} + n^2 \quad a_0 = 0$ $\qquad a_n = 1^2 + \cdots + n^2$

REALLY, WHY GENERATING FUNCTIONS?

QUESTION. How many ways to write

$$a+b+c+d=6$$

where a is even, b is a multiple of 5, c is at most 4, and d is at most 1? (a,b,c,d nonneg integers)

e.g. making a fruit basket

a	6	4	4	2	2	0	0
b	0	0	0	0	0	5	5
c	0	2	1	4	3	1	0
d	0	0	1	0	1	0	1

7 ways.

What about $a+b+c+d=100$

or $a+b+c+d=n$?

REALLY, WHY GENERATING FUNCTIONS?

QUESTION. How many ways to write

$$a+b+c+d=n$$

where a is even, b is a multiple of 5, c is at most 4, and d is at most 1? (a,b,c,d nonneg integers)

$$A(x) = 1+x^2+x^4+\cdots = \frac{1}{1-x^2}$$

$$B(x) = 1+x^5+x^{10}+\cdots = \frac{1}{1-x^5}$$

$$C(x) = 1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x}$$

$$D(x) = 1+x$$

As before, the answer is obtained by multiplying polynomials

$$A(x)B(x)C(x)D(x) = \frac{1}{1-x^2}\,\frac{1}{1-x^5}\,\frac{1-x^5}{1-x}\cdot 1+x$$

$$= \frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$$

Final answer: $n+1$ ways!