SECTION 5.4
Solving Recurrence Relations—Generating Functions
SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} - n/3$  (actually, this is first order)

Steps:  
1. Solve $q_n = 2q_{n-1}$  (general solution)
   
2. Find one particular solution $p_n$ to $p_n = 2p_{n-1} + n/3$
   guess: $p_n = mn + b$
   
3. Add $p_n + q_n$
   
4. Solve for constants
SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: \( a_n = 2a_{n-1} - \frac{n}{3}, \quad a_0 = 1 \)

1. \( q_n = 2q_{n-1} \quad \Rightarrow \quad q_n = c2^n \)

2. Guess: \( p_n = mn + b \)  
   \[ mn + b = 2(m(n-1) + b) - \frac{n}{3} \]
   \[ mn + b = 2mn - 2m + 2b - \frac{n}{3} \]
   \[ mn + b = (2m - \frac{1}{3})n + (2b - 2m) \]
   \( \Rightarrow \quad m = 2m - \frac{1}{3} \quad \Rightarrow \quad m = \frac{1}{3} \)
   \( b = 2b - 2m \quad \Rightarrow \quad b = 2m = \frac{2}{3} \)

So \( p_n = \frac{n}{3} + \frac{2}{3} \)

4. \( a_0 = c + \frac{2}{3} \)
   \( \Rightarrow \quad c = \frac{1}{3} \)
   \( a_n = (2^n + n + 2)/3 \)

3. \( a_n = p_n + q_n = c2^n + \frac{n}{3} + \frac{2}{3} \)
GENERATING FUNCTIONS

Sometimes counting problems, or recurrence relations can be solved using polynomials in a clever way.

Example: Find the number of solutions of
\[ a + b + c = 10 \]
where \(a\) is allowed to be 2, 3, or 4
\(b\) is allowed to be 3, 4, or 5
\(c\) is allowed to be 1, 3, or 4

The answer is the coefficient of \(x^{10}\) in
\[(x^2 + x^3 + x^4)(x^3 + x^4 + x^5)(x + x^3 + x^4)\]
e.g. \(2 + 5 + 3 \leftrightarrow x^2 x^5 x^3\)

This problem can be solved with a computer algebra system.
GENERATING FUNCTIONS

The generating function for the sequence $a_0, a_1, a_2, a_3, \ldots$ is

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

For example

- $a_n = 1 \iff 1, 1, 1, 1, \ldots \iff 1 + x + x^2 + x^3 + \ldots$
- $a_n = n + 1 \iff 1, 2, 3, 4, \ldots \iff 1 + 2x + 3x^2 + 4x^3 + \ldots$
- $a_n = n \iff 0, 1, 2, 3, \ldots \iff x + 2x^2 + 3x^3 + \ldots$
A generating function, as an object, is what is called a power series, that is, a formal sum:

\[ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]

These can be added, subtracted, and multiplied:

\[
\begin{align*}
f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \\
g(x) &= b_0 + b_1 x + b_2 x^2 + \cdots \\
\end{align*}
\]

\[
\begin{align*}
f(x) + g(x) &= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \cdots \\
f(x)g(x) &= a_0 b_0 + (a_1 b_0 + a_0 b_1) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots \\
\end{align*}
\]

But we never plug in numbers for \( x \), like with Taylor Series.

So generating functions should not be thought of as functions!
POWER SERIES

What about dividing?

Amazingly, yes! as long as $a_0 \neq 0$.

$\frac{1}{f(x)}$ is the generating function so that $f(x) \cdot \frac{1}{f(x)} = 1$

Example: $f(x) = 1 + x + x^2 + \ldots$

What is a power series that, when multiplied by $f(x)$ gives 1?

$(1-x)f(x) = 1 + 0x + 0x^2 + \ldots = 1 \Rightarrow \frac{1}{f(x)} = 1-x$, or $f(x) = \frac{1}{1-x}$

We say $\frac{1}{1-x}$ is the generating function for $a_n = 1$. 
EXAMPLES OF GENERATING FUNCTIONS

\[ \frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \iff \quad a_n = 1 \]

\[ \frac{1}{1+x} = 1 - x + x^2 - \cdots \quad \iff \quad a_n = (-1)^n \]

\[ \frac{1}{1-ax} = 1 + bx + b^2x^2 + \cdots \quad \iff \quad a_n = b^n \]

\[ \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots \quad \iff \quad a_n = n+1 \]

What is the generating function for \( a_n = n \)?

\[ a_n = n \iff x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2} \]

What about \( a_n = -2n \)?

\[ -2x/(1-x)^2 \]
SOLVING RECURRENCE RELATIONS
WITH GENERATING FUNCTIONS

Example: \( a_n = 2a_{n-1}, \ a_0 = 1 \)

The generating function for \( a_n \) is:
\[
f(x) = a_0 + a_1x + a_2x^2 + \ldots
\]

Using \( a_n = 2a_{n-1}, \) and \( a_0 = 1, \) we can rewrite each term of \( f(x): \)
\[
a_0 = 1
\]
\[
a_1x = 2a_0x
\]
\[
a_2x^2 = 2a_1x^2
\]
\[
a_3x^3 = 2a_2x^3
\]
\[
\vdots
\]

Add up:
\[
f(x) = 1 + 2x \ f(x)
\]

Solve for \( f(x): \)
\[
f(x) = \frac{1}{1-2x} \leftrightarrow a_n = 2^n
\]
Example: \( a_n = 2a_{n-1} - a_{n-2}, \ a_0 = 2, \ a_1 = -1 \)

Start with \( f(x) = a_0 + a_1x + a_2x^2 + \cdots \)

Then

\[
\begin{align*}
a_0 &= 2 \\
a_1x &= -x \\
a_2x^2 &= 2a_1x^2 - a_0x^2 \\
a_3x^3 &= 2a_2x^3 - a_1x^3 \\
&\vdots
\end{align*}
\]

Add up:

\[
f(x) = \left(2x f(x) + 2 - 5x\right) - x^2 f(x)
\]

\[
\Rightarrow f(x) = \frac{2 - 5x}{(1-2x-x^2)} = \frac{2}{(1-x^2)^2} - \frac{5x}{(1-x^2)^2}
\]

\[
\Rightarrow a_n = 2(n+1) - 5n = -3n + 2.
\]
PARTIAL FRACTIONS

Example: Rewrite \( \frac{1-x}{1-5x+6x^2} \) as a sum of fractions where the denominator is linear.

\[
\frac{1-x}{1-5x+6x^2} = \frac{1-x}{(1-3x)(1-2x)} = \frac{A}{1-3x} + \frac{B}{1-2x}
\]

\[\sim A(1-2x) + B(1-3x) = 1-x\]

\[x = \frac{1}{2} \sim B(1-\frac{3}{2}) = \frac{1}{2} \sim B = -1\]

\[x = \frac{1}{3} \sim A(1-\frac{2}{3}) = \frac{2}{3} \sim A = 2\]

\[
\frac{1-x}{1-5x+6x^2} = \frac{2}{1-3x} - \frac{1}{1-2x}
\]
SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: Solve \( a_n = 5a_{n-1} - 6a_{n-2} \quad a_0 = 1, a_1 = 4 \)

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \]

\[ \leadsto \begin{align*}
  a_0 &= 1 \\
  a_1 x &= 4x \\
  a_2 x^2 &= 5a_1 x^2 - 6a_0 x^2 \\
  a_3 x^3 &= 5a_2 x^3 - 6a_1 x^3 \\
  &\vdots
\end{align*} \]

Add up:

\[ f(x) = 5x f(x) - x + 1 - 6x^2 f(x) \]

\[ \leadsto f(x) = \frac{1 - x}{1 - 5x + 6x^2} = \frac{2}{1 - 3x} - \frac{1}{1 - 2x} \quad \leadsto \quad a_n = 2 \cdot 3^n - 2^n \]
SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: Solve $a_n = a_{n-1} + a_{n-2}$  \hspace{1cm} a_0 = 0, a_1 = 1

As above, get: $f(x) = \frac{x}{1 - x - x^2}$

Partial fractions: $1 - x - x^2 = (1-ax)(1-bx)$

$f(x) = \frac{1/\sqrt{5}}{1-ax} - \frac{1/\sqrt{5}}{1-bx}$

So $a_n = \frac{1}{\sqrt{5}}(a^n - b^n)$

Note: $ab = -1, a+b = 1 \hspace{1cm} a-b = \sqrt{5}$
Solving Recurrence Relations with Generating Functions and Partial Fractions

Example: \( a_n = 2a_{n-1} - \frac{n}{3}, \ a_0 = 1 \)

Example: \( a_n = a_{n-1} + n^2, \ a_0 = 0 \) \( a_n = 1^2 + \cdots + n^2 \)
REALLY, WHY GENERATING FUNCTIONS?

**Question.** How many ways to write
\[ a + b + c + d = 6 \]
where \( a \) is even, \( b \) is a multiple of 5, \( c \) is at most 4, and \( d \) is at most 1? (\( a, b, c, d \) nonneg integers)
e.g. making a fruit basket

\[
\begin{array}{c|cccccccc}
  a & 6 & 4 & 4 & 2 & 2 & 0 & 0 \\
  b & 0 & 0 & 0 & 0 & 0 & 5 & 5 \\
  c & 0 & 2 & 1 & 4 & 3 & 1 & 0 \\
  d & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]
7 ways.

What about \( a + b + c + d = 100 \)
or \( a + b + c + d = n \) ?
REALLY, WHY GENERATING FUNCTIONS?

**Question.** How many ways to write

\[ a + b + c + d = n \]

where \( a \) is even, \( b \) is a multiple of 5, \( c \) is at most 4, and \( d \) is at most 1? (\( a, b, c, d \) nonneg integers)

\[
\begin{align*}
A(x) &= 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2} \\
B(x) &= 1 + x^5 + x^{10} + \cdots = \frac{1}{1-x^5} \\
C(x) &= 1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x} \\
D(x) &= 1 + x 
\end{align*}
\]

As before, the answer is obtained by multiplying polynomials

\[
A(x)B(x)C(x)D(x) = \frac{1}{1-x^2} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^5}{1-x} \cdot (1+x)
\]

\[
= \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots
\]

Final answer: \( n+1 \) ways!