CHAPTER 6
Counting

SECTION 6.1
The inclusion-exclusion principle

A COUNTING PROBLEM

I have 60 cats.
27 of them are dumb, 42 of them are ugly.
How many are dumb and ugly?
How many are neither?

Need some rules for counting.

COUNTING RULES

Say A and B are subsets of a set U.

(a) $|A \cup B| = |A| + |B| - |A \cap B|$ ← inclusion-exclusion

(b) $|A \cap B| \leq \min\{|A|, |B|\}$

(c) $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$

(d) $|A^c| = |U| - |A|$

(e) $|A \Delta B| = |A \cup B| - |A \cap B| = |A| + |B| - 2|A \cap B| = |A \setminus B| + |B \setminus A|$

(f) $|A \times B| = |A| \cdot |B|$

(g) $|A \cap B| \geq |A| + |B| - |U|$

SOLVING THE CAT PROBLEM

U = the set of cats
A = the set of dumb cats
B = the set of ugly cats

Neither dumb nor ugly: $|A^c \cap B^c| = |(A \cup B)^c| \overset{(d)}{=} |U| - |A \cup B|$

$$\overset{(a)}{=} |U| - (|A| + |B| - |A \cap B|)$$
$$= |U| - |A| - |B| + |A \cap B|$$
$$= |A \cap B| - N$$

where $N = |A| + |B| - |U|$

So:

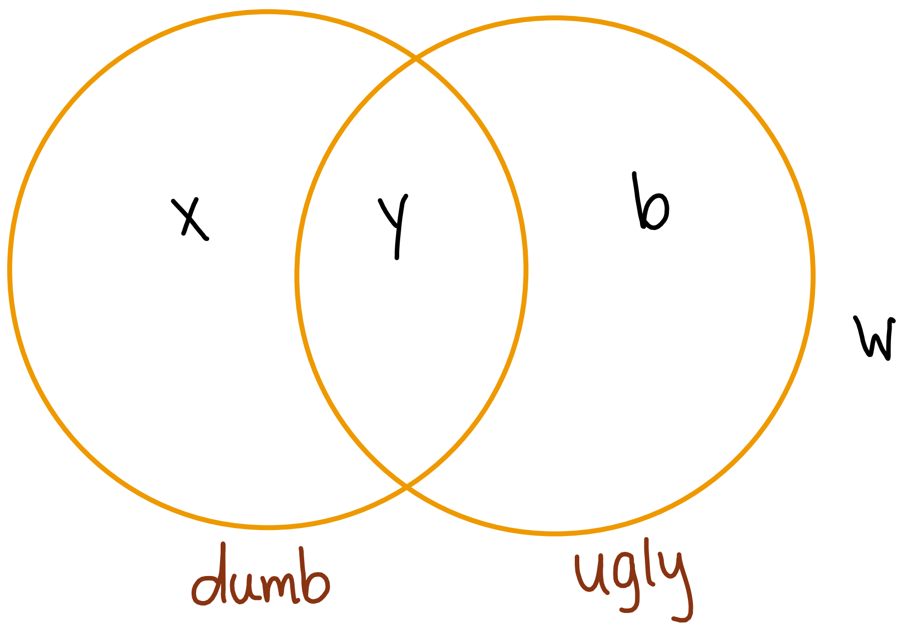
$$0 = N - N \overset{(g)}{\leq} \underset{\parallel}{|A \cap B| - N} \overset{(b)}{\leq} \min\{|A|, |B|\} - N$$

$$0 \leq |A^c \cap B^c| \leq 27 - (27 + 42 - 60) = 18$$

What about dumb and ugly?

SOLVING THE CAT PROBLEM

Could also make a system of linear (in)equalities:



$$x + y = 27$$
$$y + z = 42$$
$$x + y + z + w = 60$$

MORE PROBLEMS

1. Out of 80 cars, 20 have brakes and 15 have headlights. 10 have both. How many have either brakes or headlights? How many have neither?

2. Out of 1,000 students, 800 know French, 500 know Spanish, 325 know both. How many know neither?

THE INCLUSION-EXCLUSION PRINCIPLE

$$|A| = |A|$$
$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$
$$\vdots$$
$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$
$$+ \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

Proof: By the binomial theorem*:

$$(1+x)^n = 1^n + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Plug in $x = -1$:

$$0 = 1 - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \cdot 1$$
$$\rightsquigarrow \quad 1 = \binom{n}{1} - \binom{n}{2} + \cdots + (-1)^{n+1} \cdot \binom{n}{n}$$

RHS is # times an element is counted in inc.-exc.

* Will cover in Chapter 7.

INCLUSION-EXCLUSION PROBLEMS

1. Out of 100 students, 18 like Chik-fil-A, 40 like Taco Bell, 20 like Subway, 12 like CFA and TB, 5 like CFA and SW, 4 like TB and SW, and 3 like them all.

   How many students like at least one of these? How many don't like any? How many only like Subway.

2. How many integers between 1 and 500 (inclusive) are:

   (a) not divisible by 2? (c) divisible by 2, 3, or 7?

   (b) divisible by 2 or 3? (d) divisible by 2, 3, 5, or 7?

Key idea: Let A_k = numbers divisible by k

If $\gcd(j,k)=1$, then $A_{jk} = A_j \cap A_k$.

Section 6.2
The addition and multiplication rules

THE ADDITION RULE

The number of ways a collection of mutually exclusive events can occur is the sum of the number of ways each event can occur.

or: If $A_1, \ldots, A_n$ are pairwise disjoint sets, then

$$|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$$

Examples:

- If there are 12 choices on the breakfast menu and 15 choices on the lunch menu, there are 27 choices for your meal.
- How many ways to roll a 6 with two dice?
- How many integers between 1 and 22 are divisible by 3 or 8?

THE MULTIPLICATION RULE

The number of ways in which a sequence of independent events can occur is the product of the numbers of ways in which each event can occur.

or: $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$

What are the "events" in this formula?

Examples:
- 5 shirts, 3 pants $\Rightarrow$ 15 outfits
- Eat brunch and lunch $\Rightarrow$ $12 \cdot 15$ pairs of meals
- There are 256 bytes (= bit strings of length 8)
- Mr. Potato Head:

Mr. Potato Head was born on May 1, 1952. The original toy cost $0.98, and contained hands, feet, ears, two mouths, two pairs of eyes, four noses, three hats, eyeglasses, a pipe, and eight felt pieces resembling facial hair. The original Mr. Potato Head kit did not come with a potato "body", so parents had to provide their own potato into which children could stick the various pieces.

Walsh, Tim (2005). Timeless Toys: Classic Toys and the Playmakers Who Made Them.

THE PRINCIPLE OF OVERCOUNTING

Sometimes it is easier to count some items several times <u>or</u> count some items that shouldn't have been counted in the first place and correct yourself later.

Example: At a party, how many handshakes took place if 12 people had 5 handshakes, 9 people had 3 handshakes and 1 person had 3 handshakes?

$$\frac{12\cdot 5+9\cdot 3+1\cdot 3}{2}=45$$