SECTION 6.3
The Pigeonhole Principle

THE PIGEONHOLE PRINCIPLE

If n objects are put into m boxes, and $n > m$, then at least one box will have multiple objects.

In other words, if A and B are finite sets with $|A| > |B|$ then there is no injective function $A \to B$.

Example: In New York City, there are two people with the same number of hairs on their heads.

\# hairs on a head: hundreds of thousands
\# people in NYC: millions.

THE PIGEONHOLE PRINCIPLE

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Johann Peter Gustav Lejeune Dirichlet

PIGEONHOLE PROBLEMS

1. Show that, given 5 points in a unit square, there are two points within $\sqrt{2}/2$ of each other.

2. Show that, given any 11 integers, there is a pair of numbers whose difference is divisible by 10.

3. Show that, at any party, there are always two people with the same number of friends.

   Hint. Treat two cases: (a) Everyone has at least 1 friend
   (b) Someone has no friends

4. Take a chessboard with two opposite corners removed. Can you cover it with dominos?

   Hint: The dominos give a bijection between black squares and white squares.

PIGEONHOLE PROBLEMS

5. On a 5×5 chessboard, there is one flea in each square. Each flea jumps to an adjacent square. Are there now two fleas in the same square?

6. Arrange the numbers $1, \ldots, 10$ on a circle in any order. Show that there are 3 consecutive numbers that add to 17 or more.

THE STRONG FORM OF THE PIGEONHOLE PRINCIPLE

If n objects are put into m boxes, then some box must have at least

$$\left\lceil \frac{n}{m} \right\rceil$$

objects.

Example: If you have 8 fruits, all apples, bananas, and oranges, then there must be either 3 apples, 3 oranges, or 3 bananas.

Example: Our class has 109 students. How many were born in the same month?

The challenge with counting
is that we aren't usually told
in advance which rules to use

MORE PROBLEMS

1. How many 3 digit numbers are there?
2. How many 3 digit numbers are there with no repeated digits?
3. How many 3 digit numbers are there with the i^{th} digit equal to i for some i.
4. How many functions are there $A \to B$ if $|A| = m$, $|B| = n$?
5. How many injective functions are there $A \to B$ if $|A| = m$, $|B| = n$?
6. How many subsets of A are there if $|A| = n$?

MORE PROBLEMS

7. How many even 4 digit numbers are there with no repeated digits?

8. How many odd 4 digit numbers are there with no repeated digits? (Harder!)

9. How many ways are there to place a domino on a chessboard?

10. How many bit strings are there that have length n and begin and/or end with a 1?

11. How many different dominos are there?

12. How many arrangements are there of 6 men and 4 women at a round table if no women sit together?

MORE PROBLEMS

13. Given 20 integers, show there is a pair whose difference is divisible by 19.

14. If we want to label the chairs in a room by one letter and one number from 1 to 100, how many labels are there?

15. How many distinct alphanumeric passcodes are there if each passcode has 6-8 characters and at least one digit?

16. In how many ways can a best-of-5 series go down?

17. Given 5 points on a sphere, how many necessarily lie on the same hemisphere?