CHAPTER 7
PERMUTATIONS AND COMBINATIONS

SECTION 7.1
Permutations

Permutations

The basic question: How many ways are there to make an ordered list?

Example: In a club with 30 people, how many ways are there to choose a president, vice president, and secretary?

$30 \cdot 29 \cdot 28$

A permutation of n objects is an arrangement of those objects in some order.

How many permutations of n objects?

$n!$

r-PERMUTATIONS

An r-permutation of n objects is a choice of r of the objects and a permutation of those r objects.

How many r-permutations of n objects?

$$n!/(n-r)! = n(n-1)\cdots(n-r+1)$$

We define

$$P(n,r) = n!/(n-r)!$$

In other words, $P(n,r)$ is the number of ways to put r distinguishable marbles into n boxes, at most one marble to a box.

Example: In an olympic event with 8 athletes, in how many ways can gold, silver, and bronze be awarded?

PERMUTATION PROBLEMS

1. How many ways are there to give 7 sad puppies away to 20 people, if each person can take at most one sad puppy?

$P(20,7) = 20!/13!$

2. A group has n men and n women. In how many ways can they be lined up so that men and women alternate?

3 choices: ① an ordering of n men $n!$
② an ordering of n women $n!$
③ man first or woman first 2

Multiplication rule $\rightsquigarrow 2 \cdot n!\, n!$

PERMUTATION PROBLEMS

3. How many ways are there to seat 6 boys and 4 girls at a round table if no two girls sit together?

Note: A rotation of a configuration is considered the same as the original configuration.

overcounting

① Arrange the 6 boys in a circle: $6!/6$

② Then there are 6 slots for 4 girls, at most one per slot: $P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3$

Multiplication rule: $5!\,6!/2$

PERMUTATION PROBLEMS

4. Arrange all 26 letters of the alphabet in a row.

a) How many such "words" are there?

$26!$

b) How many contain HAMLET as a subword, e.g.:

VRPKGCHAMLETBDFIZWJNQOSYUX

Idea: Think of $\boxed{\text{HAMLET}}$ as a single, big "letter"

$\rightsquigarrow 21!$

c) How many have exactly 4 letters between H and T?

$2 \cdot (24 \cdot 23 \cdot 22 \cdot 21) \cdot 21!$

COMBINATIONS

The basic question: How many ways to make an unordered list with n items?

Example: In a club with 30 people, how many ways to choose a committee with 3 members?

$30 \cdot 29 \cdot 28 / 6$

Or: How many ways to put 3 indistinguishable marbles in 30 boxes?

MARBLES AND BOXES

Distinguishable marbles: Say we want to put a red, a green, and a blue marble into 5 boxes. How many ways?

$$P(5,3) = 5 \cdot 4 \cdot 3 = 60$$

Indistinguishable marbles: Say we want to put 3 indistinguishable marbles in 5 boxes. How many ways?

$$P(5,3)/3! = 5 \cdot 4 \cdot 3/6 = 10$$

orderings of the 3 marbles

N CHOOSE K

Number of ways to put k indistinguishable marbles in n boxes:

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!} \qquad \text{“n choose k”}$$

Fact: $\binom{n}{k} = \binom{n}{n-k}$

Proof: Choosing k objects to be in a set is the same as choosing n-k objects to be not in the set.

Can also use the formula. $\blacksquare$

COMBINATION PROBLEMS

1. Five people need a ride. My car holds 4. In how many ways can I choose who gets a ride?

$$\binom{5}{4} = \binom{5}{1} = 5$$

2. If you toss a coin 7 times, in how many ways can you get 4 heads?

Need to choose 4 of 7 "slots" to be heads.

$$\binom{7}{4} = {}^{7!}/_{3!4!} = {}^{7\cdot 6\cdot 5}/_{6} = 35$$

3. The House of Representatives has 435 representatives. How many 4-person committees can there be?

$$\binom{435}{4} = \frac{435\cdot 434\cdot 433\cdot 432}{4!} = 1{,}471{,}429{,}260$$

MORE PROBLEMS

1. How many bit strings are there with fifteen 0's and six 1's if every 1 is followed by a 0?

Note: Too hard if you think of it as a sequence of 21 tasks.

Idea: Think of arranging six 10's and nine 0's.
Choose which of the 15 slots should be the 0's:

$$\binom{15}{9} = 5005$$

Second idea: Put fifteen 0's down, then put the six 1's into the fifteen possible slots.

MORE PROBLEMS

2. How many strings in the letters a, b, and c have length 10 and exactly 4 a's?

Again, don't choose the 10 letters one by one.

First choose 4 slots for a's : $\binom{10}{4} = 210$

Then 2 choices in each remaining slot : $2^6 = 64$

$\rightsquigarrow 210 \cdot 64 = 13440$

MORE PROBLEMS

3. A lottery ticket has six numbers from 1 to 40. How many different tickets are there?

$$\binom{40}{6} = 3{,}838{,}380$$

The lottery agency chooses six winning numbers. How many different possible lottery tickets have exactly four winning numbers?

First, choose 4 out of the 6 winning numbers: $\binom{6}{4} = 15$

Then, choose 2 non-winning numbers: $\binom{34}{2} = 561$

$\rightsquigarrow$ 8415 tickets

MORE PROBLEMS

4. Determine the number of alphabetic strings of length 5 consisting of distinct (capital) letters that
   - (a) do not contain A
   - (b) contain A
   - (c) start with ABC
   - (d) start with A, B, C in any order
   - (e) contain A, B, C in that order
   - (f) contain A, B, C

5. Determine the number of possible softball teams (= 9 people) can be made from a group of 10 men, 12 women, and 17 children if:
   - (a) there are no restrictions
   - (b) there must be 3 men, 3 women, 3 children
   - (c) the team must be all men, all women, or all children
   - (d) the team cannot have both men and women.

MORE PROBLEMS

6. In how many ways can you put 5 indistiguishable red balls and 8 indistinguishable green balls into 20 boxes if
   - (a) there can be at most one ball per box
   - (b) there can be at most one ball of each color per box.

7. How many poker hands are:
   - (a) total
   - (b) 4 of a kind
   - (c) flush
   - (d) straight
   - (e) straight flush
   - (f) full house
   - (g) 3 of a kind
   - (h) 2-pair
   - (i) pair
   - (j) neither flushes straights, full house 3 of a kind, 2 pair, pair