MATH 2602

Linear and Discrete Mathematics

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WHAT IS DISCRETE MATH?

dis·crete [dih-**skreet**] ? Show IPA

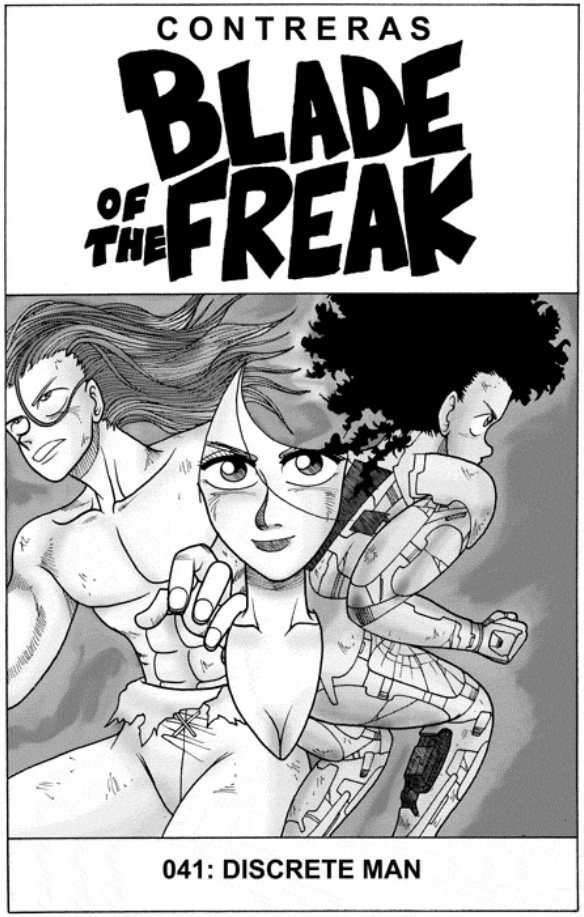
adjective

1. apart or detached from others; separate; distinct: *six discrete parts.*
2. consisting of or characterized by distinct or individual parts; discontinuous.
3. *Mathematics* .
   a. (of a topology or topological space) having the property that every subset is an open set.
   b. defined only for an isolated set of points: *a discrete variable.*
   c. using only arithmetic and algebra; not involving calculus: *discrete methods.*

dictionary.com

Discrete is the opposite of continuous.

WHAT IS DISCRETE MATH?



WHAT IS DISCRETE MATH?

CONTINUOUS	DISCRETE
real numbers	integers
measuring	counting
ideal shapes	computer images
wave	particle
differential eqn	recurrence reln.
calculus	probability graph theory algorithms

CHAPTER 5

INDUCTION & RECURSION

Section 5.1

Mathematical Induction

Towers of Hanoi

Proposition: The Towers of Hanoi puzzle with n disks is solvable.

Proof: First, the puzzle is easily solvable when $n = 1$.

Second, if I have a solution to the puzzle with n disks, then I can easily turn that into a solution for the puzzle with $n+1$ disks.

Therefore, I conclude that the puzzle is solvable for $n = 1, 2, 3, \ldots$ disks!

DOMINOS

If I knock the first domino down, and I know each falling domino causes the next one to fall, then I know that <u>all</u> the dominos will fall (even if there are infinitely many).

CLICKER QUESTION

How cute is this kid?

a) very cute

b) extremely cute

c) ridiculously cute

d) cutest kid I have ever seen

THE PRINCIPLE OF MATHEMATICAL INDUCTION

Say we have a mathematical statement that depends on a natural number n. Suppose:

1. The statement is true for $n = n_0$.
2. Whenever the statement is true for $n = k$, it is true for $n = k+1$.

Then the statement is true for all $n \geq n_0$.

HOW TO DO A PROOF BY INDUCTION

First, prove the proposition for a <u>base case</u> $n = n_0$

Next, assume the proposition is true for $n = k$

<u>Using</u> the assumption, prove that the proposition is true for $n = k+1$.

By the principle of mathematical induction, conclude that the proposition is true for all $n \geq n_0$.

EXAMPLE

PROPOSITION: For $n \geq 1$:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

PROOF: The proposition is true for $n = 1$:

$$1 = \frac{1(1+1)}{2}$$

Assume the proposition is true for $n = k$, that is:

$$1 + \cdots + k = \frac{k(k+1)}{2}$$

Using the assumption, we will show the proposition is true for $n = k+1$:

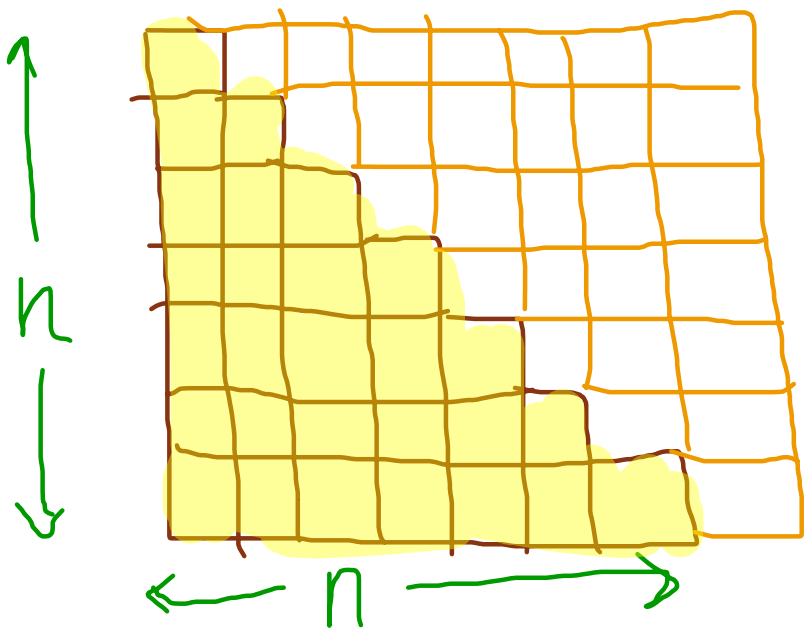
$$
\begin{aligned}
1 + \cdots + k + 1 &= (1 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + k + 1 \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}
\end{aligned}
$$

By the principal of mathematical induction, the proposition is proven. $\blacksquare$

Two Non-Inductive Proofs

$$
\begin{array}{rccccccc}
 & 1 & 2 & 3 & \cdots & n-1 & n \\
+ & n & n-1 & n-2 & \cdots & 2 & 1 \\
\hline
 & n+1 & n+1 & n+1 & & n+1 & n+1
\end{array}
$$

Total: $n(n+1)$ First row: $\frac{n(n+1)}{2}$



Area: $\frac{n(n+1)}{2}$

EXAMPLE

PROPOSITION: $7^n - 1$ is divisible by 6 for $n \geq 0$.

PROOF: First, the proposition is true for $n=0$:

$$7^0 - 1 = 1 - 1 = 0$$

and 0 is divisible by 6.

Now, assume the proposition is true for $n=k$:

$$7^k - 1 \text{ is divisible by } 6.$$

Using the assumption, we'll show the proposition is true for $n=k+1$, that is:

$$7^{k+1} - 1 \text{ is divisible by } 6.$$

$$
\begin{aligned}
7^{k+1} - 1 &= 7 \cdot 7^k - 1 \\
&= 7 \cdot 7^k - 7 + 6 \\
&= 7(7^k - 1) + 6
\end{aligned}
$$

divisible by 6 by our assumption.

The sum of two numbers divisible by 6 is again divisible by 6.

By the principle of mathematical induction, the the proposition is proven. $\blacksquare$

More Examples

Proposition: Any debt of $n \geq 4$ dollars can be paid with \$2 bills and \$5 bills.

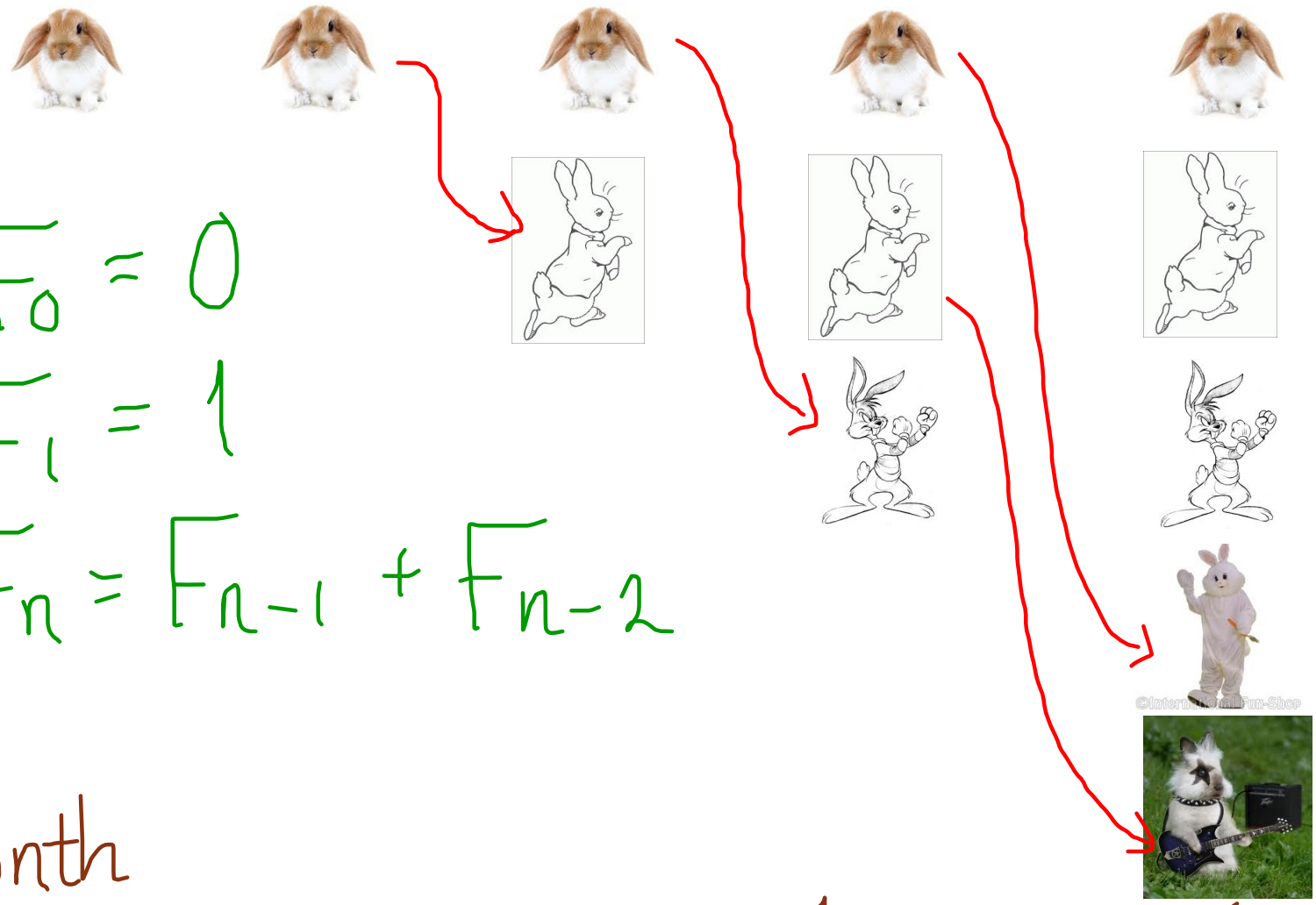
Proposition: For $n \geq 1$:

$$\frac{1}{n+1} + \cdots + \frac{1}{2n} \geq \frac{1}{2}$$

Hint: Use +/- trick again.

Proposition: Any n lines in the plane with no two parallel and no triple intersections divide the plane into $n(n+1)/2 + 1$ regions.

Fibonacci Numbers



$F_0 = 0$

$F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$

Month				
1	2	3	4	5

An Induction Problem with Fibonacci Numbers

Proposition: For $n \geq 1$:

$$F_1 + \cdots + F_n = F_{n+2} - 1$$

Proof: First, the proposition is clearly true for $n = 1$:

$$F_1 = 1 = F_3 - 1 = 2 - 1$$

Next, we assume the proposition is true for $n = k$:

$$F_1 + \cdots + F_k = F_{k+2} - 1$$

Now, using the assumption, we show the proposition is true for $n = k+1$:

$$F_1 + \cdots + F_{k+1} = F_{k+1}$$

$$
\begin{aligned}
F_1 + \cdots + F_{k+1} &= (F_1 + \cdots F_k) + F_{k+1} \\
&= F_{k+2} - 1 + F_{k+1} \\
&= (F_{k+1} + F_{k+2}) - 1 \\
&= F_{k+3} - 1 \\
&= F_{(k+1)+2} - 1
\end{aligned}
$$

By the principle of mathematical induction, the proposition is proven. $\blacksquare$

Leonardo Fibonacci

More Problems

Proposition: For $n \geq 0$:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Proposition: For $n \geq 1$:

$$F_1 + \cdots + F_{2n-1} = F_{2n}$$

Proposition: The number of n-digit binary strings with no consecutive 1's is F_{n+2}.

5.1 MATHEMATICAL INDUCTION

THE PRINCIPLE OF MATHEMATICAL INDUCTION

Say we have a mathematical statement that depends on a natural number n. Suppose:

1. The statement is true for $n = n_0$.
2. Whenever the statement is true for $n = k$, it is true for $n = k+1$.

Then the statement is true for all $n \geq n_0$.

EXAMPLE

PROPOSITION: For $n \geq 0$ $\quad (1 + 1/2)^n \geq 1 + \frac{n}{2}$

PROOF: First we check the base case $n = 0$:

$$1 = (1 + 1/2)^0 \geq 1 + 0/2 = 1$$

Next, we assume the proposition is true for $n = k$:

$$(1 + 1/2)^k \geq 1 + k/2$$

Using the assumption, we prove the proposition for $n = k+1$:

$$\begin{aligned}(1 + 1/2)^{k+1} = (1 + 1/2)(1 + 1/2)^k &\geq (1 + 1/2)(1 + k/2) \\ &= 1 + k/2 + 1/2 + k/4 \\ &\geq 1 + (k/2 + 1/2) \\ &= 1 + (k+1)/2\end{aligned}$$

By the principle of mathematical induction, the proposition is proven. $\blacksquare$

THE STRONG FORM OF THE PRINCIPAL OF MATHEMATICAL INDUCTION

Say we have a mathematical statement that depends on a natural number n. Suppose that

1. The statement is true for $n = n_0$
2. Whenever the statement is true for all natural numbers in the interval $[n_0, k]$, then it is also true for $n = k+1$.

Then the statement is true for all $n \geq n_0$.

Note: It may be that more than one base case is needed! The number of base cases needed is dictated by the inductive argument.

EXAMPLE

PROPOSITION: Every natural number $n \geq 2$ is a product of prime numbers.

PROOF: The base case $n=2$ is obviously true.
Now, assume that every natural number n in $[2, k-1]$ is a product of prime numbers. We must show that k is a product of prime numbers.

First, if k is prime, there is nothing to do. On the other hand, if k is not prime, it is equal to a product $k = mn$, where $2 \leq m, n < k$. By our inductive hypothesis, both m and n are products of prime numbers. Therefore, k is itself a product of prime numbers. $\blacksquare$

EXAMPLE

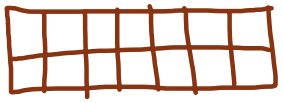
PROPOSITION: The number of ways of breaking a $2\times n$ candy bar into 2×1 bars is F_{n+1}.

PROOF: First we check the base cases $n=1$ and $n=2$:


only 1 way, and $F_2=1$.


two ways, and $F_3=2$

We assume the proposition is true for $1,\dots,k-1$, where $k\geq 3$. We must now prove the proposition for $n=k$:



There are two ways to break off the end: one vertical piece, or two 2×1 horizontal pieces.
In the 1st case we get a $2\times(k-1)$ bar $\rightsquigarrow F_k$ ways.
2nd case $\rightsquigarrow 2\times(k-2)$ bar $\rightsquigarrow F_{k-1}$ ways
In total, $F_{k-1}+F_k=F_{k+1}$ ways to break the $2\times k$ bar.
By strong induction, the proposition is proven.

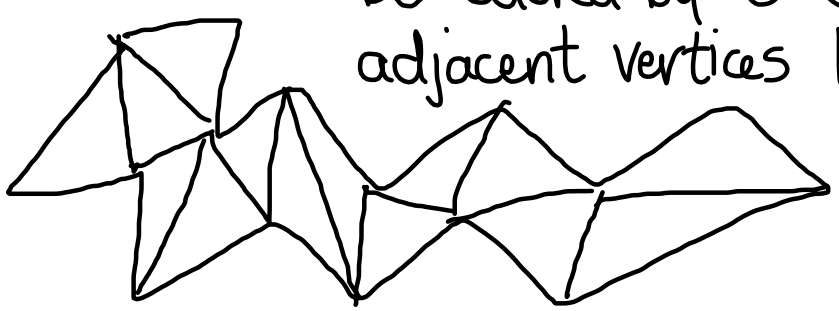
MORE EXAMPLES

Consider the sequence $a_1 = 1$, $a_2 = 2$, $a_3 = 3$
$a_k = a_{k-1} + a_{k-2} + a_{k-3}$ for $k \geq 4$.

PROPOSITION: $a_n < 2^n$ for all $n > 0$.

PROPOSITION: In a regular n-gon, one can draw at most $n-3$ diagonals that do not cross.

PROPOSITION: The vertices of a triangulated n-gon can always be colored by 3 colors so that no two adjacent vertices have the same color.



SECTION 5.2

RECURRENCE RELATIONS

RECURRENCE RELATIONS

A <u>recurrence</u> <u>relation</u> for a sequence $(a_n)_{n=0}^{\infty}$ is an equation expressing each term a_n in terms of its predecessors $a_1, \ldots, a_{n-1}$.

If some a_i are given specific values, those are called <u>initial</u> <u>conditions</u>.

A first example:

$a_n = a_{n-1} + 3, \quad a_0 = 0.$

Solution: $a_n = 3n$

Like a differential eqn:

$f'(x) = 3 \qquad f(0) = 0$

Solution: $f(x) = 3x$

EXAMPLES OF RECURRENCE RELATIONS

	Closed form	Recursive form
Exponentials	$a_n = 2^n$	$a_n = 2a_{n-1}$, $a_0 = 1$
Factorials	$a_n = n!$	$a_n = n a_{n-1}$, $a_0 = 1$
Arithmetic seq.	$a_n = dn + b$	$a_n = a_{n-1} + d$, $a_0 = b$
Geometric seq.	$a_n = cr^n$	$a_n = r a_{n-1}$, $a_0 = c$

↳ e.g. Money market account:
Put in \$500, collect 7% annually
$a_n = 500(1.07)^n$

MORE EXAMPLES

Annuity: Deposit \$200/yr, get 7% interest/year

$a_n = 1.07 \cdot a_{n-1} + 200$

closed form?

Wilhelm Ackermann

Fibonacci numbers: $a_0 = 0,\ a_1 = 1$

$a_n = a_{n-1} + a_{n-2}$

closed form?

Ackermann Function:

(i) $A(n,0) = A(n-1,1) \quad n = 1,2,\ldots$

(ii) $A(n,k) = A(n-1, A(n,k-1)) \quad n,k = 1,2,\ldots$

(iii) $A(0,k) = k+1 \quad k = 0,1,\ldots$

Very hard to compute:

$A(0,0)=1,\ A(1,1)=3,\ A(2,2)=7,\ A(3,3)=61$

MORE EXAMPLES

Annuity: Deposit \$200/yr, get 7% interest/year

$a_n = 1.07 \cdot a_{n-1} + 200$

closed form?

Fibonacci numbers: $a_0 = 0,\ a_1 = 1$

$a_n = a_{n-1} + a_{n-2}$

closed form?

Wilhelm Ackermann

Ackermann Function:

(i) $A(n,0) = A(n-1,1) \quad n = 1,2,\ldots$

(ii) $A(n,k) = A(n-1, A(n,k-1)) \quad n,k = 1,2,\ldots$

(iii) $A(0,k) = k+1 \quad k = 0,1,\ldots$

Very hard to compute:

$A(0,0)=1,\ A(1,1)=3,\ A(2,2)=7,\ A(3,3)=61$

$A(4,4) = 2^{2^{2^{2^{2}}}} - 3$ $\longrightarrow$ over 10^{19199} digits

universe has $\sim 10^{80}$ elementary particles

SOLVING RECURRENCE RELATIONS

A <u>solution</u> to a recurrence relation is an explicit formula for the sequence.

Example: Consider the arithmetic sequence

$$a_0 = -2, \quad a_n = a_{n-1} + 5.$$

Solution:

$$a_n = 5n - 2$$

More generally: $a_0 = b, \; a_n = a_{n-1} + m$

Solution:

$$a_n = mn + b$$

Can prove by induction.

SOLVING RECURRENCE RELATIONS

Example: Consider

$$a_0 = 7, \quad a_n = -3a_{n-1}$$

Solution:

$$a_n = 7(-3)^n$$

More generally: Consider

$$a_0 = c, \quad a_n = ra_{n-1}$$

Solution:

$$a_n = ra_{n-1}$$

A MORE INTERESTING EXAMPLE

Example: Solve the recurrence relation

$$a_0 = 0, \quad a_n = 2a_{n-1} + 1$$

Note: this formula describes the number of moves in our solution to the Towers of Hanoi puzzle.

Let's find the first few terms:

$$a_0 = 0$$
$$a_1 = 1$$
$$a_2 = 3$$
$$a_3 = 7$$
$$a_4 = 15$$
$$a_5 = 31$$

Guess: $a_n = 2^n - 1$

VERIFYING OUR GUESS

PROPOSITION: The solution to

$$a_0 = 0, \quad a_n = 2a_{n-1} + 1$$

is $a_n = 2^n - 1$.

PROOF: We proceed by induction on n.

Base case $n = 0$: $a_0 = 0 = 2^0 - 1$ ✓

Assume the proposition is true for $n = k$:

$$a_k = 2^k - 1$$

Using the assumption, we show the proposition is true for $n = k+1$:

$$\begin{aligned} a_{k+1} &= 2a_k + 1 \\ &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

✓ ∎

MORE COMPLICATED RECURSION RELATIONS

What about $a_0 = 1, \quad a_n = 2a_{n-1} + 3$?

or $a_0 = 1, \; a_1 = 2, \; a_n = 2a_{n-1} + 3a_{n-2}$?

SECTION 5.3

Solving Recurrence Relations: The Characteristic Polynomial

WHY STUDY RECURRENCE RELATIONS?

Reason #1: Sometimes a sequence of numbers is more easily described this way, e.g.: the number of moves in our solution to the Towers of Hanoi problem is $a_n = 2a_{n-1} + 1$

Also, the number of Fibonacci rabbits: $a_n = a_{n-1} + a_{n-2}$

Reason #2: They are discrete versions of differential equations:

$$a_n' = a_n - a_{n-1} \qquad a_n'' = a_n' - a_{n-1}'$$

So differential equations can be approximated by a difference equation, then converted to a recurrence relation.

SOLVING RECURRENCE RELATIONS

To <u>solve</u> a recurrence relation means to give an explicit formula.

Example: $a_n = a_{n-1} + 2, \quad a_0 = 1$

Solution: $a_n = 2n + 1$

Can use induction to <u>prove</u> this is a solution:

Base case: $a_0 = 1 = 2 \cdot 0 + 1$

Assume: $a_k = 2k + 1$

Show $a_{k+1} = 2(k+1) + 1$:

$$\begin{aligned} a_{k+1} &= a_k + 2 \\ &= (2k+1) + 2 \\ &= 2(k+1) + 1 \quad \checkmark \end{aligned}$$

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

$$a_n = r a_{n-1} + s a_{n-2}$$

Second order: a_n defined in terms of a_{n-1}, a_{n-2}

Linear: A <u>linear</u> combination of x and y is

$$5x - 2y$$

not

$$5xy \quad \text{or} \quad e^x \quad \text{or} \quad \sqrt{x+y}$$

Homogeneous: No "extra stuff" after the linear combination of a_{n-1} and a_{n-2}.

Extra stuff = function of n.

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} + a_{n-2}$, $a_0 = 0, a_1 = 1$

What is the solution?

First few terms: $0, 1, 2, 5, 12, 29, 70, 169, \ldots$

What is the pattern?

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

It turns out we can solve them all!

<u>Theorem</u>: Consider the recurrence relation

$$a_n = ra_{n-1} + sa_{n-2}.$$

Let b_1, b_2 be the roots of

$$x^2 - rx - s$$

Then the solution to a_n is:

$$a_n = \begin{cases} c_1 b_1^n + c_2 b_2^n & \text{if } b_1 \neq b_2 \\ c_1 b_1^n + c_2 n b_2^n & \text{if } b_1 = b_2 \end{cases}$$

The c_i are determined by the initial conditions.

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = a_{n-2}$, $a_0 = 1$, $a_1 = 3$.

We can write this as: $a_n = 0 \cdot a_{n-1} + a_{n-2}$

$\rightsquigarrow x^2 - 0 \cdot x - 1 = x^2 - 1 = (x+1)(x-1)$

So $b_1 = 1$, $b_2 = -1$

By the theorem:

$$a_n = c_1(1)^n + c_2(-1)^n$$
$$= c_1 + c_2(-1)^n$$

Find c_1, c_2 using initial conditions:

$$a_0 = 1 = c_1 + c_2$$
$$a_1 = 3 = c_1 - c_2$$

$\rightsquigarrow c_1 = 2,\ c_2 = -1$

$\rightsquigarrow a_n = 2 + (-1)(-1)^n = 2 + (-1)^{n+1}$

SECOND ORDER HOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1, a_1 = 0$

$\rightsquigarrow x^2 - 6x + 9 \rightsquigarrow (x-3)^2 \rightsquigarrow b_1 = b_2 = 3$

$\rightsquigarrow a_n = c_1 3^n + c_2 n 3^n$

Use the initial conditions to find the c_i:

$$a_0 = c_1 = 1$$
$$a_1 = 3c_1 + 3c_2 = 3 + 3c_2 = 3(1 + c_2) = 0$$

$\rightsquigarrow c_1 = 1,\ c_2 = -1$

So: $a_n = 3^n - n3^n$

THE CASE $b_1 = b_2$

$$b_1 = b_2$$

$$\iff (x-b_1)^2 = x^2 - 2b_1 x + b_1^2$$

$$\iff a_n = 2b_1 a_{n-1} - b_1^2 a_{n-2}$$

$$\iff a_n = r a_{n-1} + s a_{n-2}$$

where $s = -r^2/4$

MORE PROBLEMS

① Solve $a_n = 9a_{n-2}$ where

(a) $a_0 = 6,\ a_1 = 12$

(b) $a_0 = 6,\ a_2 = 54$

(c) $a_0 = 6,\ a_2 = 10$

② Solve $a_n = 8a_{n-1} - 16a_{n-2},\ a_0 = 1,\ a_1 = 16$

③ Solve $5a_n = 11a_{n-1} - 2a_{n-2},\ a_0 = 2,\ a_1 = -8.$

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

General form: $a_n = ra_{n-1} + sa_{n-2} + f(n)$

Examples:

$$a_n = 2a_{n-1} + 1$$
$$a_n = 3a_{n-1} + 2a_{n-2} + n$$
$$a_n = 5a_{n-1} - a_{n-2} + 2^n$$
$$a_n = a_{n-1} + a_{n-2} + (n^7 + n^n + n!)$$

We do not know how to solve them all, but...

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

THEOREM: Let $a_n = ra_{n-1} + sa_{n-2} + f(n)$.
Let p_n be any particular solution to a_n.
Let q_n be the general solution to $q_n = rq_{n-1} + sq_{n-2}$.
Then $p_n + q_n$ is the general solution to a_n.

We already have a sure-fire way to find q_n.

The hard part is that we don't know how to find p_n — we have to <u>guess</u>.

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

THEOREM: Let $a_n = ra_{n-1} + sa_{n-2} + f(n)$.
Let p_n be any particular solution to a_n.
Let q_n be the general solution to $q_n = rq_{n-1} + sq_{n-2}$.
Then $p_n + q_n$ is the general solution to a_n.

Proof that $p_n + q_n$ really is a solution:

By definition: $p_n = rp_{n-1} + sp_{n-2} + f(n)$
$q_n = rq_{n-1} + sq_{n-2}$

Let $t_n = p_n + q_n$. Adding the last two lines:

$$t_n = rt_{n-1} + st_{n-2} + f(n) \checkmark$$

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

EXAMPLE: Solve $a_n = 2a_{n-1} + 1$

First we solve $q_n = 2q_{n-1}$

$\rightsquigarrow x^2 - 2x \rightsquigarrow x = 0, 2$

$\rightsquigarrow x = k2^n$

Then we find a particular solution to a_n by "guessing":

$a_n = -1$

Check: $-1 = 2 \cdot (-1) + 1$ ✓

By the theorem, the general solution is:

$a_n = k2^n - 1$

We find k using initial conditions.

HOW TO GUESS PARTICULAR SOLUTIONS

If $f(n)$ is...	Guess p_n to be...
exponential	exponential (same base)
linear	linear
quadratic	quadratic
n^{th} degree polynomial	n^{th} degree polynomial
anything else	???

Example: Solve $a_n = 3a_{n-1} + 5\cdot 7^n$, $a_0 = 2$.

First we "guess" p_n:

$$p_n = c7^n$$

Need to find c: $c7^n = 3c7^{n-1} + 5\cdot 7^n$

$$7^{n-1}(7c - 3c - 5\cdot 7) = 0$$

$$c = 35/4$$

$$\rightsquigarrow p_n = 35/4\, 7^n = 5/4\, 7^{n+1}$$

Then we solve $q_n = 3q_{n-1} \rightsquigarrow q_n = k3^n$

By the theorem $a_n = p_n + q_n = k3^n + 5/4\, 7^{n+1}$

Now we find k: $2 = a_0 = k + 35/4$

$$k = -27/4$$

$$\rightsquigarrow a_n = -27/4 \cdot 3^n + 5/4\, 7^{n+1} = -\frac{1}{4}3^{n+3} + \frac{5}{4}7^{n+1}$$

Example: $a_n = -a_{n-1} + n$, $a_0 = 1/4$.

First we guess $p_n = mn + b$ Need to find m, b:

$$\begin{aligned} mn + b &= -(m(n-1)+b) + n \\ &= -(mn - m + b) + n \\ &= -mn + m - b + n \\ &= (1-m)n + (m-b) \end{aligned}$$

$\rightsquigarrow m = 1/2, \quad b = 1/4$

$\rightsquigarrow p_n = \frac{1}{2}n + \frac{1}{4}$

Then we solve $q_n = -q_{n-1} \rightsquigarrow q_n = k(-1)^n$

By the theorem: $a_n = k(-1)^n + (1/2\, n + 1/4)$

Using initial condition: $a_0 = 1/4 = k + 1/4 \rightsquigarrow k = 0$

So: $a_n = n/2 + 1/4$.

MORE PROBLEMS

① Solve $a_n = 5a_{n-1} - 6a_{n-2} + 6 \cdot 4^n$

② Solve $a_n = a_{n-1} + 3n^2$, $a_0 = 7$

By the way, there is another method for solving #2, the method of undetermined coefficients. Idea: recursively substitute: $a_n = a_0 + \sum_{i=1}^{n} f(i) = 7 + 3\sum i^2 = \cdots$

SECTION 5.4
SOLVING RECURRENCE RELATIONS – GENERATING FUNCTIONS

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} - n/3$ (actually, this is first order)

Steps:

1. Solve $q_n = 2q_{n-1}$ (general solution)
2. Find one particular solution p_n to $p_n = 2p_{n-1} + n/3$
   guess: $p_n = mn + b$
3. Add $p_n + q_n$
4. Solve for constants

SECOND ORDER NONHOMOGENEOUS LINEAR RECURRENCE RELATIONS

Example: $a_n = 2a_{n-1} - n/3, \quad a_0 = 1$

① $q_n = 2q_{n-1}$

$\rightsquigarrow q_n = c2^n$

② Guess: $p_n = mn + b$ Need to find m, b

$$mn + b = 2(m(n-1) + b) - n/3$$
$$mn + b = 2mn - 2m + 2b - n/3$$
$$mn + b = \left(2m - \tfrac{1}{3}\right)n + (2b - 2m)$$

$\rightsquigarrow m = 2m - 1/3 \rightsquigarrow m = 1/3$

$b = 2b - 2m \rightsquigarrow b = 2m = 2/3$

So $p_n = n/3 + 2/3$

③ $a_n = p_n + q_n = c2^n + n/3 + 2/3$

④ $a_0 = c + 2/3$

$\rightsquigarrow c = 1/3$

$a_n = (2^n + n + 2)/3$

GENERATING FUNCTIONS

Sometimes counting problems, or recurrence relations can be solved using polynomials in a clever way.

Example: Find the number of solutions of

$$a+b+c=10$$

where a is allowed to be $2, 3,$ or 4
b is allowed to be $3, 4,$ or 5
c is allowed to be $1, 3,$ or 4

The answer is the coefficient of x^{10} in

$$(x^2+x^3+x^4)(x^3+x^4+x^5)(x+x^3+x^4)$$

e.g. $2+5+3 \longleftrightarrow x^2x^5x^3$

This problem can be solved with a computer algebra system.

GENERATING FUNCTIONS

The <u>generating</u> <u>function</u> for the sequence

$$a_0, a_1, a_2, a_3, \ldots$$

is

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

For example

$$a_n = 1 \longleftrightarrow 1, 1, 1, 1, \ldots \longleftrightarrow 1 + x + x^2 + x^3 + \cdots$$

$$a_n = n + 1 \longleftrightarrow 1, 2, 3, 4, \ldots \longleftrightarrow 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$a_n = n \longleftrightarrow 0, 1, 2, 3, \ldots \longleftrightarrow x + 2x^2 + 3x^3 + \cdots$$

POWER SERIES

A generating function, as an object, is what is called a power series, that is, a formal sum ← Think "string"

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

These can be added, subtracted, and multiplied:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$
$$f(x)g(x) = a_0 b_0 + (a_1 b_0 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

But we never plug in numbers for x, like with Taylor Series.

So generating functions should not be thought of as functions!

POWER SERIES

What about dividing?

Amazingly, yes! as long as $a_0 \neq 0$.

$\frac{1}{f(x)}$ is the generating function so that $f(x) \cdot \frac{1}{f(x)} = 1$

Example: $f(x) = 1 + x + x^2 + \cdots$

What is a power series that, when multiplied by $f(x)$ gives 1?

$(1-x)f(x) = 1 + 0x + 0x^2 + \cdots = 1 \rightsquigarrow \frac{1}{f(x)} = 1-x$, or $f(x) = \frac{1}{1-x}$

We say $1/1-x$ is the generating function for $a_n = 1$.

EXAMPLES OF GENERATING FUNCTIONS

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \longleftrightarrow \quad a_n = 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots \quad \longleftrightarrow \quad a_n = (-1)^n$$

$$\frac{1}{1-ax} = 1 + bx + b^2x^2 + \cdots \quad \longleftrightarrow \quad a_n = b^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots \quad \longleftrightarrow \quad a_n = n+1$$

* The other three follow from the first one by substitution & differentiation

What is the generating function for $a_n = n$?

$$a_n = n \longleftrightarrow x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}$$

What about $a_n = -2n$?

$$-2x/(1-x)^2$$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS

Example: $a_n = 2a_{n-1}$, $a_0 = 1$

The generating function for a_n is:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

Using $a_n = 2a_{n-1}$, and $a_0 = 1$, we can rewrite each term of $f(x)$:

$$a_0 = 1$$
$$a_1 x = 2a_0 x$$
$$a_2 x^2 = 2a_1 x^2$$
$$a_3 x^3 = 2a_2 x^3$$
$$\vdots$$

Add up: $f(x) = 1 + 2x\, f(x)$

Solve for $f(x)$:

$$f(x) = 1/1-2x \longleftrightarrow a_n = 2^n$$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS

Example: $a_n = 2a_{n-1} - a_{n-2}$, $a_0 = 2$, $a_1 = -1$

Start with $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

Then

$$a_0 = 2$$
$$a_1 x = -x$$
$$a_2 x^2 = 2a_1 x^2 \qquad - a_0 x^2$$
$$a_3 x^3 = 2a_2 x^3 \qquad - a_1 x^3$$
$$\vdots$$

Add up: $f(x) = \left(2x f(x) + 2 - 5x\right) - x^2 f(x)$

$$\rightsquigarrow f(x) = \frac{2-5x}{(1-2x-x^2)} = \frac{2}{(1-x^2)^2} - \frac{5x}{(1-x^2)^2}$$

$$\rightsquigarrow a_n = 2(n+1) - 5n = -3n + 2.$$

PARTIAL FRACTIONS

Example: Rewrite $\frac{1-x}{1-5x+6x^2}$ as a sum of fractions where the denominator is linear.

$$\frac{1-x}{1-5x-6x^2} = \frac{1-x}{(1-3x)(1-2x)} = \frac{A}{1-3x} + \frac{B}{1-2x}$$

$$\rightsquigarrow A(1-2x) + B(1-3x) = 1-x$$

$$x = 1/2 \rightsquigarrow B\left(1-\tfrac{3}{2}\right) = \tfrac{1}{2} \rightsquigarrow B = -1$$

$$x = 1/3 \rightsquigarrow A(1-2/3) = 2/3 \rightsquigarrow A = 2$$

$$\frac{1-x}{1-5x+6x^2} = \frac{2}{1-3x} - \frac{1}{1-2x}$$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: Solve $a_n = 5a_{n-1} - 6a_{n-2}$ $\quad a_0 = 1, a_1 = 4$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\rightsquigarrow \quad \begin{aligned} a_0 &= 1 \\ a_1 x &= 4x \\ a_2 x^2 &= 5a_1 x^2 - 6a_0 x^2 \\ a_3 x^3 &= 5a_2 x^3 - 6a_1 x^3 \\ &\vdots \end{aligned}$$

Add up: $\quad f(x) = 5x f(x) - x + 1 - 6x^2 f(x)$

$$\rightsquigarrow f(x) = \frac{1-x}{1-5x+6x^2} = \frac{2}{1-3x} - \frac{1}{1-2x} \rightsquigarrow a_n = 2 \cdot 3^n - 2^n$$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: Solve $a_n = a_{n-1} + a_{n-2}$ $\quad a_0 = 0,\ a_1 = 1$

As above, get: $f(x) = \dfrac{x}{1-x-x^2}$

$a = \frac{1+\sqrt{5}}{2},\ b = \frac{1-\sqrt{5}}{2}$

Partial fractions: $1-x-x^2 = (1-ax)(1-bx)$

Note: $ab = -1,\ a+b = 1$

$a-b = \sqrt{5}$

$$f(x) = \frac{1/\sqrt{5}}{1-ax} - \frac{1/\sqrt{5}}{1-bx}$$

So $a_n = \frac{1}{\sqrt{5}}\left(a^n - b^n\right)$

SOLVING RECURRENCE RELATIONS WITH GENERATING FUNCTIONS AND PARTIAL FRACTIONS

Example: $a_n = 2a_{n-1} - n/3, \quad a_0 = 1$

Example: $a_n = a_{n-1} + n^2 \quad a_0 = 0$ $\qquad a_n = 1^2 + \cdots + n^2$

REALLY, WHY GENERATING FUNCTIONS?

QUESTION. How many ways to write

$$a+b+c+d=6$$

where a is even, b is a multiple of 5, c is at most 4, and d is at most 1? (a,b,c,d nonneg integers)

e.g. making a fruit basket

a	6	4	4	2	2	0	0
b	0	0	0	0	0	5	5
c	0	2	1	4	3	1	0
d	0	0	1	0	1	0	1

7 ways.

What about $a+b+c+d=100$

or $a+b+c+d=n$?

REALLY, WHY GENERATING FUNCTIONS?

QUESTION. How many ways to write

$$a+b+c+d=n$$

where a is even, b is a multiple of 5, c is at most 4, and d is at most 1? (a,b,c,d nonneg integers)

$$A(x) = 1+x^2+x^4+\cdots = \frac{1}{1-x^2}$$
$$B(x) = 1+x^5+x^{10}+\cdots = \frac{1}{1-x^5}$$
$$C(x) = 1+x+x^2+x^3+x^4 = \frac{1-x^5}{1-x}$$
$$D(x) = 1+x$$

As before, the answer is obtained by multiplying polynomials

$$A(x)B(x)C(x)D(x) = \frac{1}{1-x^2}\,\frac{1}{1-x^5}\,\frac{1-x^5}{1-x}\cdot 1+x$$
$$= \frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$$

Final answer: $n+1$ ways!

SECTION 8.1
ALGORITHMS

ALGORITHMS

Algorithm: A clearly specified method (or procedure for solving a problem.

Etymology: Abu Ja'far Muḥammad ibn Mûsâ al-Khwârizmî

PROBLEMS

We distinguish between a problem and an instance of a problem.

PROBLEM	INSTANCE
Matrix multiplication	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
Traveling salesman	What is the most efficient route for my mail carrier?
Sudoku	

COMPLEXITY

Given an algorithm, we can ask about its cost.

The cost will be a function of the size of the input.
e.g. multiplying two numbers.

Complexity $f: \mathbb{N} \longrightarrow \mathbb{R}$

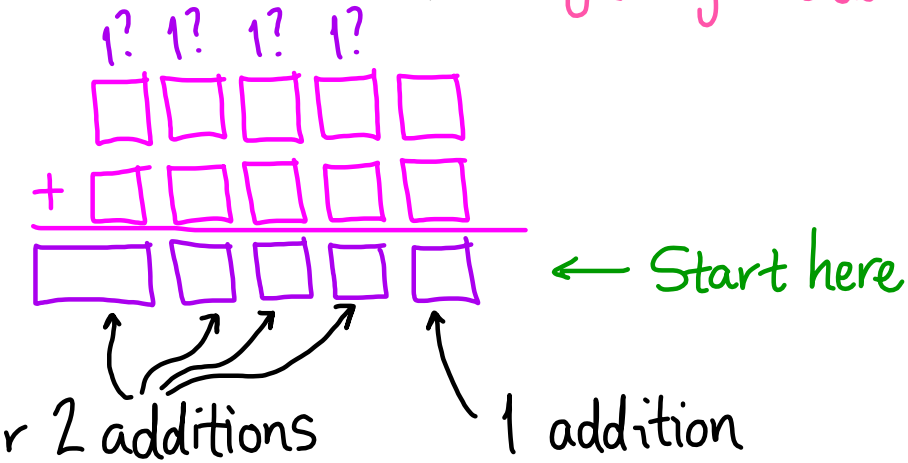
size of input $\rightsquigarrow$ cost for running the algorithm on input of that size. (worst case scenario)

For this to make sense, we need to specify what we mean by size and cost. Our answer will depend on the particular problem, as well as our particular needs.

COMPLEXITY

Example: What is the cost of adding two n digit numbers in terms of the number of single digit additions?

Answer:



$\rightsquigarrow f(n) = 1 + 2(n-1) = 2n-1$

THE BIG QUESTION

Is there a better way to do things?

Given two algorithms, which is more efficient?

Given one algorithm for a problem, is there a better one out there?

What do we mean by better?

$$7n^2 - 52 \sim n^2$$

but $n < n^2$

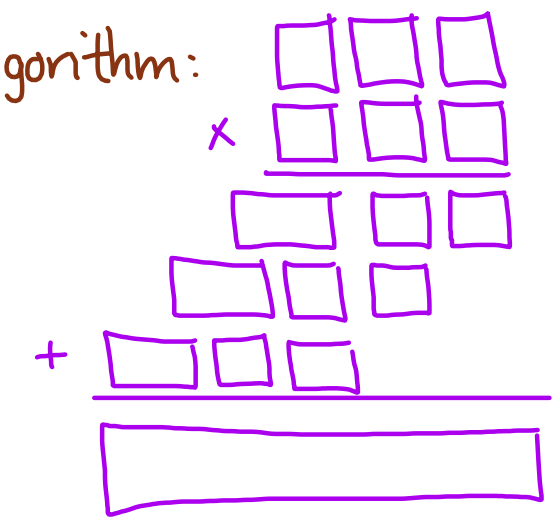
better than

EXAMPLE: MULTIPLYING TWO NUMBERS

Problem: What is the complexity of multiplying two n digit numbers in terms of the number of single digit multiplications?

(What about additions??)

Grade school algorithm:



$f(n) = n^2$

Is there a better way?

DIVIDE AND CONQUER

Idea: Break the problem into more manageable subproblems.

This usually leads to a recursive relation for the complexity, since the complexity of the bigger problem is given in terms of the complexity of the smaller problems.

Is there such an algorithm for multiplying two numbers?

MULTIPLYING TWO NUMBERS

Divide and Conquer Algorithm: The idea is to break up both n digit numbers into two $n/2$ digit numbers and multiply those:

$$a = \boxed{a_1 \mid a_2} = a_1 \cdot 10^{n/2} + a_2$$
$$b = \boxed{b_1 \mid b_2} = b_1 \cdot 10^{n/2} + b_2$$

$$ab = a_1 b_1 10^n + [a_1 b_2 + a_2 b_1] 10^{n/2} + a_2 b_2$$

Example: $1011 \cdot 1213 = (10 \cdot 12) \cdot 10^4 + (130 + 132) \cdot 10^2 + 143$
$= 1200000 + 26200 + 143 = 1226343.$

Complexity: $f(n) = 4 f(n-1)$

Actually, can improve from 4 to 3, since $(a_1 b_2 + a_2 b_1) = (a_1 + a_2)(b_1 + b_2) - a_1 b_1 - a_2 b_2$

$\rightsquigarrow f(n) = 3 f(n-1).$

MULTIPLYING TWO NUMBERS

So we find the complexity of the divide and conquer algorithm by solving the recurrence relation

$$a_n = 3a_{n/2}$$

We solve this recursion relation by working backward (see last page of Lecture 3).

Assume here $n = 2^k$: $a_n = 3a_{n/2} = 3^2 a_{n/4} = \cdots = 3^k a_1 = 3^k$

$$\rightsquigarrow a_n = 3^{\log_2 n} = n^{\log_2 3}$$

This is <u>better</u> than n^2!

COMPARISON

#digits	# multiplications: Grade school algorithm	# multiplications: Divide & conquer algorithm	factor
10	100	39	2.5
100	10,000	1479	6.76
1,000	1,000,000	56,871	17.58
10,000	100,000,000	2,187,007	45.72
100,000	10,000,000,000	84,103,197	118.90
1,000,000	1×10^{12}	3,234,260,557	309.19

MORE EXAMPLES

① Matrix multiplication

Usual algorithm: n^2 multiplications
Divide & conquer: $n^{\log_2 7}$

Idea: Divide matrices into 4 submatrices, then do 7 multiplications.

② Evaluation of polynomials

Usual algorithm: $2n-1$ multiplications

Horner's method: Write $p(x)$ as $x \cdot q(x) + c$
$\rightsquigarrow$ recursive relation for # of multiplications:
$f(n) = f(n-1) + 1 \rightsquigarrow f(n) = n$

③ Greatest common divisor

④ Searching a list

A MILLION DOLLAR PROBLEM

A problem is of type P if it has a polynomial solution.

A problem is of type NP if, handed a solution to an instance of the problem, there is a polynomial time algorithm to check if it really is a solution.

e.g. factoring

QUESTION: P = NP?

If P = NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in "creative leaps," no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss...
— Scott Aaronson, MIT

NP-COMPLETE PROBLEMS

There is a (huge!) list of NP problems that "contain" all other NP problems, including:

Sudoku	Minesweeper
Battleship	Free Cell
Tetris	etc.

To prove P=NP, show any one of these problems has a polynomial solution.

To prove P≠NP, show any one of these problems has no polynomial solution.

SECTION 8.2
Complexity

BIG O

O for "order" (of magnitude)

Let f and g be functions $\mathbb{N} \to \mathbb{R}$.
We say that "f is big O of g" and write

$$f = O(g) \quad \text{or} \quad f \in O(g)$$

if there is a natural number n_0 and a positive real number c such that

$$|f(n)| \leq c\,|g(n)|$$

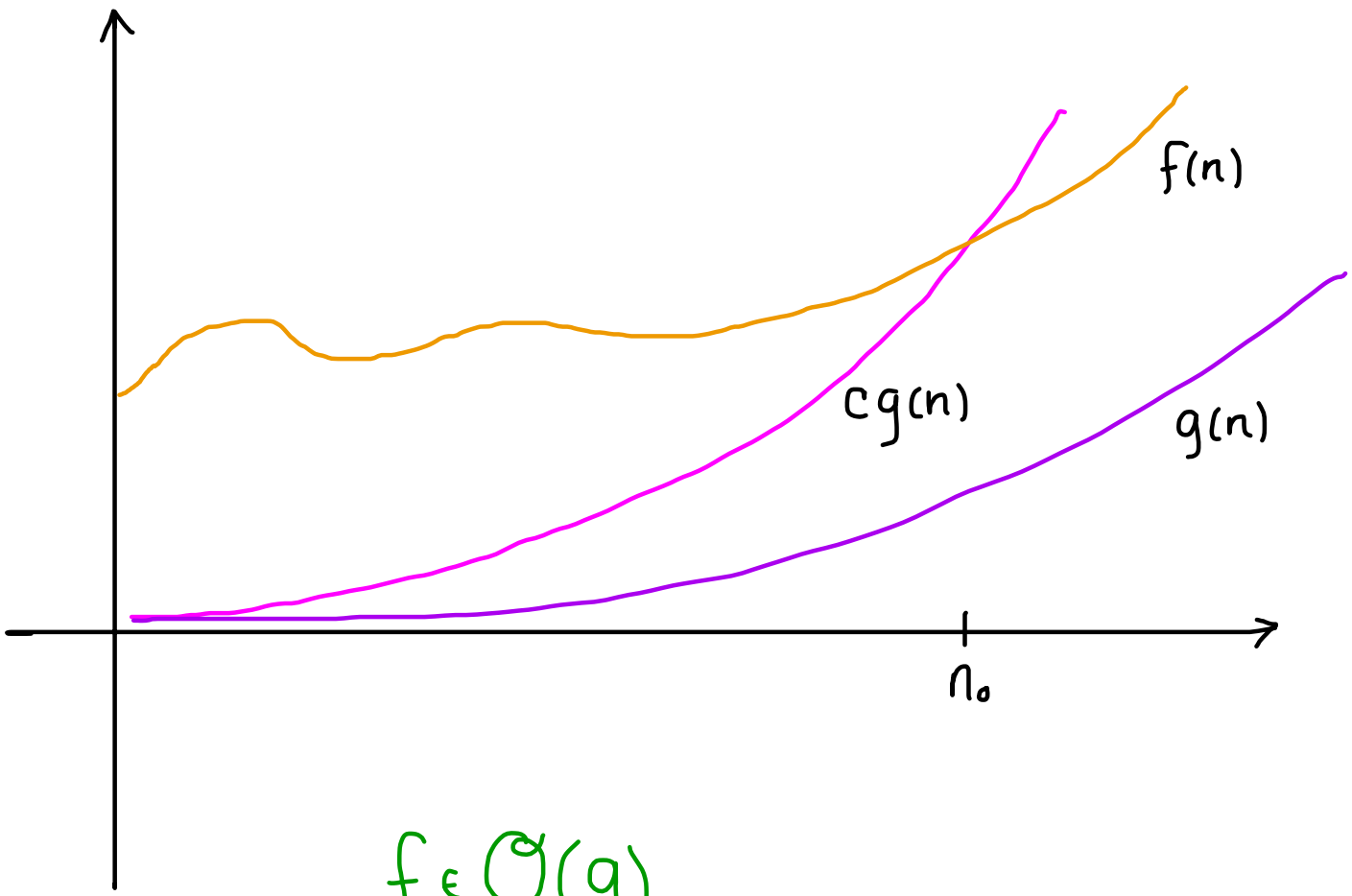
for $n \geq n_0$. "for large n"

Note: If $f, g : \mathbb{N} \to [0, \infty)$ we can drop the absolute values.

Note: There are infinitely many choices for n_0 and c.

Observation: If $f(n) \leq g(n)$ for all n, then f is $O(g)$

BIG O



$f \in O(g)$

BIG O

We say that "f is big O of g" and write

$$f = O(g) \quad \text{or} \quad f \in O(g)$$

if there is a natural number n_0 and a positive real number c such that

$$|f(n)| \leq c\,|g(n)|$$

for $n \geq n_0$.

First examples: ① $f(n) = n^2$, $g(n) = 7n^2$

$f \in O(g)$ $\quad c = 1,\ n_0 = 1$

$g \in O(f)$ $\quad c = 7,\ n_0 = 1$

② $f(n) = 4n+2$, $g(n) = n$

$f \in O(g)$ $\quad c = 5,\ n_0 = 2$

$g \in O(f)$ $\quad c = 1,\ n_0 = 1$

ANOTHER EXAMPLE

Example: $f(n) = n^2$, $g(n) = n^2 + n$

$f \in O(g)$ $\quad c = 1, n_0 = 1$

$g \in O(f)$?

Want $n^2 + n \leq cn^2$ for n large

$(c-1)n^2 \geq n$

$(c-1)n \geq 1$

$\rightsquigarrow c = 2, n_0 = 1$

So $g \in O(f)$.

We say f and g have the same order.

NOT BIG O

How do we show f is <u>not</u> $O(g)$?

Need to show no c, n_0 work.

Example: $f(n) = n \quad g(n) = \sqrt{n}$

First, g <u>is</u> $O(f)$: $c = 1$, $n_0 = 1$

But, is it possible that $n \le c\sqrt{n}$ for large n ($n \ge n_0$)?
This would mean $\sqrt{n} \le c$ for large n.
Impossible!

We conclude f is <u>not</u> $O(g)$.

COMPARING FUNCTIONS

Let f and g be functions $\mathbb{N} \to \mathbb{R}$.

We say...	and write...	if...
f has smaller order than g	$f < g$	$f \in O(g)$ $g \notin O(f)$
f has the same order as g	$f \asymp g$	$f \in O(g)$ $g \in O(f)$

MORE EXAMPLES

Show that $5n^3+12n \asymp n^3$

Clearly $n^3 \in O(5n^3+12n)$
Also, $5n^3+12n \leq 6n^3$ for $n \geq 4$
$\rightsquigarrow 5n^3+12n \in O(n^3)$.

Show that $n+1 \asymp n$

MORE EXAMPLES

① Compare $n!$ & n^n

② Compare $n!$ & 2^n

COMBINING FUNCTIONS

Theorem: Let f, g be functions $\mathbb{N} \to \mathbb{R}$.

(a) If $f \in \mathcal{O}(F)$, then $f + F \in \mathcal{O}(F)$

(b) If $f \in \mathcal{O}(F)$ and $g \in \mathcal{O}(G)$ then $fg \in \mathcal{O}(FG)$.

Proof: (a)

$$\begin{aligned} |f(n) + F(n)| &\leq |f(n)| + |F(n)| \\ &\leq c|F(n)| + |F(n)| && n \geq n_0 \\ &= (c+1)|F(n)| && n \geq n_0 \end{aligned}$$

$\blacksquare$

For example, $(n+1)(5n^3 + 12n) = 5n^4 + 5n^3 + 12n^2 + 12n$ is $\mathcal{O}(n^4)$ by (b)

What about $19n^{58} + n^{18} - 3n^{10}$?

$\asymp n^{58}$ by (a).

BIG O VIA LIMITS

THEOREM: Let f, g be functions $\mathbb{N} \to [0, \infty)$

(a) If $\lim_{n\to\infty} f(n)/g(n) = 0$, then $f < g$

(b) If $\lim_{n\to\infty} f(n)/g(n) = \infty$, then $g < f$

(c) If $\lim_{n\to\infty} f(n)/g(n) = L \neq 0$, then $f \asymp g$

PROOF: (a) $\lim f(n)/g(n) = 0$ means: For all $\varepsilon > 0$, there exists n_0 so that $|f(n)/g(n)| < \varepsilon$ when $n \geq n_0$. In other words

$$|f(n)| < \varepsilon |g(n)| \quad , \quad n \geq n_0 \qquad (*)$$

$\rightsquigarrow f \in O(g)$

On the other hand, need $g \neq O(f)$.

$g = O(f)$ means $|g(n)| \leq c|f(n)| \quad n \geq n_0$

i.e. $\frac{1}{c}|g(n)| \leq |f(n)| \quad n \geq n_0$

contradicting $(*)$ $\blacksquare$

POLYNOMIALS

Theorem: Let $f(n) = a_d n^d + \cdots a_1 n + a_0$ be a degree d polynomial ($a_d \neq 0$). Then $f(n) \asymp n^d$.

Can prove using either of the last two theorems.

Proof:

$$\lim_{n\to\infty} \left| f(n)/g(n) \right| = \lim_{n\to\infty} \left| \frac{a_d n^d + \cdots + a_0}{n^d} \right|$$

$$= \lim_{n\to\infty} \left| \frac{a_d + a_{d-1}/n + \cdots + a_1/n^{d-1} + a_0/n^d}{1} \right|$$

$$= |a_d|. \qquad \blacksquare$$

MORE COMPARISONS

Theorem: (a) If $k < \ell$, then $n^k < n^\ell$

(b) If $k > 1$, then $\log_k n < n$

(c) If $k > 0$, then $n^k < 2^k$

Proof: (b) $$\lim_{n\to\infty} \frac{\log_k n}{n} = \lim_{n\to\infty} \frac{\ln n}{\ln k\, n} \overset{L'H}{=} \lim_{n\to\infty} \frac{1/n}{\ln k \cdot 1} = 0.$$

Apply the limit theorem. $\blacksquare$

HIERARCHY

$$1 < \log n < n < n^k < k^n < n! < n^n$$

const < log < linear < poly < exp < fact < tower

MORE DETAILED HIERARCHY

$$1 < \log n < \sqrt{n} < n/\log n < n < n\log n < n^{3/2}$$

$$< n^2 < n^3 < \cdots$$

$$< 2^n < 3^n < \cdots$$

$$< n!$$

$$< n^n < n^{n^n} < \cdots$$

COMPARING DIFFERENT ORDERS

	10	50	100	300	1000
$5n$	50	250	500	1500	5,000
$n \log n$	33	282	665	2469	9966
n^2	100	2500	10,000	90,000	1,000,000
n^3	1,000	125,000	1 mil	27 mil	1 bil
2^n	10^{24}	16 digits	31 dig.	91 dig.	302 dig.
$n!$	3.6 mil	65 dig.	161 dig.	623 dig.	unimaginable
n^n	10 bil.	85 dig.	201 dig.	744 dig.	unimaginable

μsecs since big bang: $\sim 10^{24}$

protons in the known universe: $\sim 10^{126}$

D. Harel, Algorithmics

COMPARING DIFFERENT ORDERS

How long would it take at 1 step per μsec?

	10	20	50	100	300
n^2	1/10,000 sec.	1/2500 sec.	1/400 sec	1/100 sec.	9/100 sec.
n^5	1/10 sec.	3.2 sec	5.2 min	2.8 hr	28.1 days
2^n	1/1,000 sec	1 sec	35.7 yr	400 trillion cent.	75 digit # of centuries
n^n	2.8 hr	3.3 trillion yr	70 digit # of centuries	185 digit # of centuries	728 digit # of centuries.

D. Harel, Algorithmics

CHAPTER 6
Counting

SECTION 6.1
The inclusion-exclusion principle

A COUNTING PROBLEM

I have 60 cats.
27 of them are dumb, 42 of them are ugly.
How many are dumb and ugly?
How many are neither?

Need some rules for counting.

COUNTING RULES

Say A and B are subsets of a set U.

(a) $|A \cup B| = |A| + |B| - |A \cap B|$ ← inclusion-exclusion

(b) $|A \cap B| \leq \min\{|A|, |B|\}$

(c) $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$

(d) $|A^c| = |U| - |A|$

(e) $|A \triangle B| = |A \cup B| - |A \cap B| = |A| + |B| - 2|A \cap B| = |A \setminus B| + |B \setminus A|$

(f) $|A \times B| = |A| \cdot |B|$

(g) $|A \cap B| \geq |A| + |B| - |U|$

SOLVING THE CAT PROBLEM

U = the set of cats
A = the set of dumb cats
B = the set of ugly cats

Neither dumb nor ugly: $|A^c \cap B^c| = |(A \cup B)^c| \overset{(d)}{=} |U| - |A \cup B|$

$$\overset{(a)}{=} |U| - (|A| + |B| - |A \cap B|)$$
$$= |U| - |A| - |B| + |A \cap B|$$
$$= |A \cap B| - N$$

where $N = |A| + |B| - |U|$

So:

$$0 = N - N \overset{(g)}{\leq} |A \cap B| - N \overset{(b)}{\leq} \min\{|A|, |B|\} - N$$

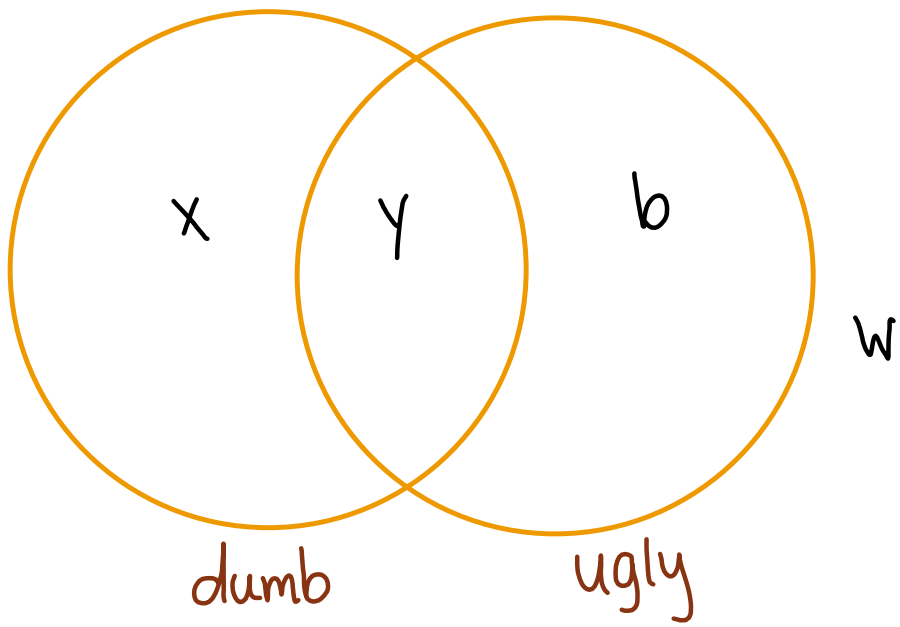
$$\parallel$$

$$0 \leq |A^c \cap B^c| \leq 27 - (27 + 42 - 60) = 18$$

What about dumb and ugly?

SOLVING THE CAT PROBLEM

Could also make a system of linear (in)equalities:



$$x + y = 27$$
$$y + z = 42$$
$$x + y + z + w = 60$$

MORE PROBLEMS

1. Out of 80 cars, 20 have brakes and 15 have headlights. 10 have both. How many have either brakes or headlights? How many have neither?

2. Out of 1,000 students, 800 know French, 500 know Spanish, 325 know both. How many know neither?

THE INCLUSION-EXCLUSION PRINCIPLE

$$|A| = |A|$$
$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$
$$\vdots$$
$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

Proof: By the binomial theorem*:

$$(1+x)^n = 1^n + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Plug in $x = -1$:

$$0 = 1 - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \cdot 1$$
$$\rightsquigarrow \quad 1 = \binom{n}{1} - \binom{n}{2} + \cdots + (-1)^{n+1} \cdot \binom{n}{n}$$

RHS is # times an element is counted in inc.-exc.

* Will cover in Chapter 7.

INCLUSION-EXCLUSION PROBLEMS

1. Out of 100 students, 18 like Chik-fil-A, 40 like Taco Bell, 20 like Subway, 12 like CFA and TB, 5 like CFA and SW, 4 like TB and SW, and 3 like them all.

   How many students like at least one of these? How many don't like any? How many only like Subway.

2. How many integers between 1 and 500 (inclusive) are:

   (a) not divisible by 2?
   (b) divisible by 2 or 3?
   (c) divisible by 2, 3, or 7?
   (d) divisible by 2, 3, 5, or 7?

Key idea: Let A_k = numbers divisible by k

If $\gcd(j,k)=1$, then $A_{jk} = A_j \cap A_k$.

SECTION 6.2
The addition and multiplication rules

THE ADDITION RULE

The number of ways a collection of mutually exclusive events can occur is the sum of the number of ways each event can occur.

or: If $A_1, \ldots, A_n$ are pairwise disjoint sets, then

$$|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$$

Examples:

- If there are 12 choices on the breakfast menu and 15 choices on the lunch menu, there are 27 choices for your meal.
- How many ways to roll a 6 with two dice?
- How many integers between 1 and 22 are divisible by 3 or 8?

THE MULTIPLICATION RULE

The number of ways in which a sequence of independent events can occur is the product of the numbers of ways in which each event can occur.

or: $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$

What are the "events" in this formula?

Examples:
- 5 shirts, 3 pants $\Rightarrow$ 15 outfits
- Eat brunch and lunch $\Rightarrow$ $12 \cdot 15$ pairs of meals
- There are 256 bytes (= bit strings of length 8)
- Mr. Potato Head:

Mr. Potato Head was born on May 1, 1952. The original toy cost $0.98, and contained hands, feet, ears, two mouths, two pairs of eyes, four noses, three hats, eyeglasses, a pipe, and eight felt pieces resembling facial hair. The original Mr. Potato Head kit did not come with a potato "body", so parents had to provide their own potato into which children could stick the various pieces.

Walsh, Tim (2005). Timeless Toys: Classic Toys and the Playmakers Who Made Them.

THE PRINCIPLE OF OVERCOUNTING

Sometimes it is easier to count some items several times or count some items that shouldn't have been counted in the first place and correct yourself later.

Example: At a party, how many handshakes took place if 12 people had 5 handshakes, 9 people had 3 handshakes and 1 person had 3 handshakes?

$$\frac{12 \cdot 5 + 9 \cdot 3 + 1 \cdot 3}{2} = 45$$

SECTION 6.3
The Pigeonhole Principle

THE PIGEONHOLE PRINCIPLE

If n objects are put into m boxes, and $n > m$, then at least one box will have multiple objects.

In other words, if A and B are finite sets with $|A| > |B|$ then there is no injective function $A \to B$.

Example: In New York City, there are two people with the same number of hairs on their heads.

hairs on a head: hundreds of thousands
people in NYC: millions.

THE PIGEONHOLE PRINCIPLE

If n objects are put into m boxes, and $n > m$, then at least one box will have multiple objects.

Johann Peter Gustav Lejeune Dirichlet

PIGEONHOLE PROBLEMS

1. Show that, given 5 points in a unit square, there are two points within $\sqrt{2}/2$ of each other.

2. Show that, given any 11 integers, there is a pair of numbers whose difference is divisible by 10.

3. Show that, at any party, there are always two people with the same number of friends.

   Hint. Treat two cases: (a) Everyone has at least 1 friend
   (b) Someone has no friends

4. Take a chessboard with two opposite corners removed. Can you cover it with dominos?

   Hint: The dominos give a bijection between black squares and white squares.

PIGEONHOLE PROBLEMS

5. On a 5×5 chessboard, there is one flea in each square. Each flea jumps to an adjacent square. Are there now two fleas in the same square?

6. Arrange the numbers $1, \ldots, 10$ on a circle in any order. Show that there are 3 consecutive numbers that add to 17 or more.

THE STRONG FORM OF THE PIGEONHOLE PRINCIPLE

If n objects are put into m boxes, then some box must have at least

$$\left\lceil \frac{n}{m} \right\rceil$$

objects.

Example: If you have 8 fruits, all apples, bananas, and oranges, then there must be either 3 apples, 3 oranges, or 3 bananas.

Example: Our class has 109 students. How many were born in the same month?

The challenge with counting is that we aren't usually told in advance which rules to use

MORE PROBLEMS

1. How many 3 digit numbers are there?

2. How many 3 digit numbers are there with no repeated digits?

3. How many 3 digit numbers are there with the $i^{\underline{th}}$ digit equal to i for some i.

4. How many functions are there $A \to B$ if $|A| = m$, $|B| = n$?

5. How many injective functions are there $A \to B$ if $|A| = m$, $|B| = n$?

6. How many subsets of A are there if $|A| = n$?

MORE PROBLEMS

7. How many even 4 digit numbers are there with no repeated digits?

8. How many odd 4 digit numbers are there with no repeated digits? (Harder!)

9. How many ways are there to place a domino on a chessboard?

10. How many bit strings are there that have length n and begin and/or end with a 1?

11. How many different dominos are there?

12. How many arrangements are there of 6 men and 4 women at a round table if no women sit together?

MORE PROBLEMS

13. Given 20 integers, show there is a pair whose difference is divisible by 19.

14. If we want to label the chairs in a room by one letter and one number from 1 to 100, how many labels are there?

15. How many distinct alphanumeric passcodes are there if each passcode has 6-8 characters and at least one digit?

16. In how many ways can a best-of-5 series go down?

17. Given 5 points on a sphere, how many necessarily lie on the same hemisphere?

CHAPTER 7
PERMUTATIONS AND COMBINATIONS

SECTION 7.1
Permutations

Permutations

The basic question: How many ways are there to make an ordered list?

Example: In a club with 30 people, how many ways are there to choose a president, vice president, and secretary?

$30 \cdot 29 \cdot 28$

A permutation of n objects is an arrangement of those objects in some order.

How many permutations of n objects?

$n!$

r - Permutations

An r-permutation of n objects is a choice of r of the objects and a permutation of those r objects.

How many r-permutations of n objects?

$$n!/(n-r)! = n(n-1)\cdots(n-r+1)$$

We define

$$P(n,r) = n!/(n-r)!$$

In other words, $P(n,r)$ is the number of ways to put r distinguishable marbles into n boxes, at most one marble to a box.

Example: In an olympic event with 8 athletes, in how many ways can gold, silver, and bronze be awarded?

PERMUTATION PROBLEMS

1. How many ways are there to give 7 sad puppies away to 20 people, if each person can take at most one sad puppy?

$P(20,7) = 20!/13!$

2. A group has n men and n women. In how many ways can they be lined up so that men and women alternate?

3 choices: ① an ordering of n men $\quad n!$
② an ordering of n women $\quad n!$
③ man first or woman first $\quad 2$

Multiplication rule $\rightsquigarrow 2 \cdot n!\,n!$

PERMUTATION PROBLEMS

3. How many ways are there to seat 6 boys and 4 girls at a round table if no two girls sit together?

Note: A rotation of a configuration is considered the same as the original configuration.

① Arrange the 6 boys in a circle: $6!/6$ ← overcounting

② Then there are 6 slots for 4 girls, at most one per slot: $P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3$

Multiplication rule: $5!\,6!/2$

PERMUTATION PROBLEMS

4. Arrange all 26 letters of the alphabet in a row.

a) How many such "words" are there?

$26!$

b) How many contain HAMLET as a subword, e.g.:
VRPKGCHAMLETBDFIZWJNQOSYUX

Idea: Think of $\boxed{\text{HAMLET}}$ as a single, big "letter"
$\rightsquigarrow 21!$

c) How many have exactly 4 letters between H and T?

$2 \cdot (24 \cdot 23 \cdot 22 \cdot 21) \cdot 21!$

COMBINATIONS

The basic question: How many ways to make an unordered list with n items?

Example: In a club with 30 people, how many ways to choose a committee with 3 members?

$$30 \cdot 29 \cdot 28 / 6$$

Or: How many ways to put 3 indistinguishable marbles in 30 boxes?

MARBLES AND BOXES

Distinguishable marbles: Say we want to put a red, a green, and a blue marble into 5 boxes. How many ways?

$$P(5,3) = 5 \cdot 4 \cdot 3 = 60$$

Indistinguishable marbles: Say we want to put 3 indistinguishable marbles in 5 boxes. How many ways?

$$P(5,3)/3! = 5 \cdot 4 \cdot 3 / 6 = 10$$

orderings of the 3 marbles

N CHOOSE K

Number of ways to put k indistinguishable marbles in n boxes:

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$$ "n choose k"

Fact: $\binom{n}{k} = \binom{n}{n-k}$

Proof: Choosing k objects to be in a set is the same as choosing n-k objects to be not in the set.

Can also use the formula. ∎

COMBINATION PROBLEMS

1. Five people need a ride. My car holds 4. In how many ways can I choose who gets a ride?

$$\binom{5}{4} = \binom{5}{1} = 5$$

2. If you toss a coin 7 times, in how many ways can you get 4 heads?

Need to choose 4 of 7 "slots" to be heads.

$$\binom{7}{4} = 7!/3!4! = 7\cdot6\cdot5/6 = 35$$

3. The House of Representatives has 435 representatives. How many 4-person committees can there be?

$$\binom{435}{4} = \frac{435\cdot434\cdot433\cdot432}{4!} = 1{,}471{,}429{,}260$$

MORE PROBLEMS

1. How many bit strings are there with fifteen 0's and six 1's if every 1 is followed by a 0?

Note: Too hard if you think of it as a sequence of 21 tasks.

Idea: Think of arranging six 10's and nine 0's.
Choose which of the 15 slots should be the 0's:
$\binom{15}{9} = 5005$

Second idea: Put fifteen 0's down, then put the six 1's into the fifteen possible slots.

MORE PROBLEMS

2. How many strings in the letters a, b, and c have length 10 and exactly 4 a's?

Again, don't choose the 10 letters one by one.

First choose 4 slots for a's : $\binom{10}{4} = 210$

Then 2 choices in each remaining slot : $2^6 = 64$

$\rightsquigarrow 210 \cdot 64 = 13440$

MORE PROBLEMS

3. A lottery ticket has six numbers from 1 to 40. How many different tickets are there?

$\binom{40}{6} = 3{,}838{,}380$

The lottery agency chooses six winning numbers. How many different possible lottery tickets have exactly four winning numbers?

First, choose 4 out of the 6 winning numbers: $\binom{6}{4} = 15$
Then, choose 2 non-winning numbers: $\binom{34}{2} = 561$
$\rightsquigarrow$ 8415 tickets

MORE PROBLEMS

4. Determine the number of alphabetic strings of length 5 consisting of distinct (capital) letters that
   (a) do not contain A
   (b) contain A
   (c) start with ABC
   (d) start with A, B, C in any order
   (e) contain A, B, C in that order
   (f) contain A, B, C

5. Determine the number of possible softball teams (= 9 people) can be made from a group of 10 men, 12 women, and 17 children if:
   (a) there are no restrictions
   (b) there must be 3 men, 3 women, 3 children
   (c) the team must be all men, all women, or all children
   (d) the team cannot have both men and women.

MORE PROBLEMS

6. In how many ways can you put 5 indistiguishable red balls and 8 indistinguishable green balls into 20 boxes if
   (a) there can be at most one ball per box
   (b) there can be at most one ball of each color per box.

7. How many poker hands are:
   (a) total
   (b) 4 of a kind
   (c) flush
   (d) straight
   (e) straight flush
   (f) full house
   (g) 3 of a kind
   (h) 2-pair
   (i) pair
   (j) neither flushes straights, full house 3 of a kind, 2 pair, pair

SECTION 7.3
Elementary Probability

INTUITIVE PROBABILITY

What is the probability that...

a) A flipped coin comes up heads?

$1/2$

b) A rolled die comes up 3?

$1/6$

c) A rolled pair of dice comes up 4?

$3/36 = 1/12$

DEFINITIONS

An experiment is a procedure that yields one of a given set of outcomes.

The sample space of the experiment is the set of possible outcomes.

$$S = \text{finite set}$$


Blaise Pascal


Pierre Laplace

An event is a subset of the sample space:

$$A \subseteq S$$

The probability of an event A, assuming each outcome of the experiment is equally likely, is:

$$P(A) = |A| / |S|$$

EXAMPLES OF EXPERIMENTS

Experiment	Sample space S	Outcome A	Probability P(A)
Flipping a coin	$\{H, T\}$	$\{H\}$	$1/2$
Rolling a die	$\{1,2,3,4,5,6\}$	$\{3\}$	$1/6$
Rolling a pair of dice	$\{(1,1),(1,2),\ldots (6,5),(6,6)\}$	$\{(1,3),(2,2),(3,1)\}$	$3/36 = 1/12$

MORE EXAMPLES

1. You toss a coin 5 times. What is the probability of getting 4 heads?

$|S| = 2^5 = 32$

$A = \{HHHHT, HHHTH, \ldots, THHHH\}$

$P(A) = 5/32.$

2. What is the probability of correctly guessing the winners in a 64-team single elimination tournament?
(Assume every team has a 50% chance of winning each game.

$|S| = 2^{63} = 18{,}446{,}744{,}073{,}709{,}551{,}616$

$P(A) = 1/2^{63}$

Population of Earth: 7 billion.

MORE EXAMPLES

3. An urn has 4 red balls, 3 green balls. You pull one ball at random. What is the probability of pulling a green ball?

$S \neq \{R, G\}$

$S = \{R1, R2, R3, R4, G1, G2, G3\}$

$A = \{G1, G2, G3\}$

$\rightsquigarrow P(A) = 3/7$

Suppose you pull one ball, replace it, then pull another ball. What is the probability of pulling two balls of the same color?

$|S| = 49$

$|A| = 9 + 16 = 25$

$\rightsquigarrow P(A) = 25/49$

MORE EXAMPLES

Same urn (4 red, 3 green). Now suppose you pull one ball, <u>don't</u> replace it, and pull another ball. What is the probability of getting two balls of the same color?

First way: pull one at a time

$$|S| = 7 \cdot 6 = 42$$

$$|A| = 3 \cdot 2 + 4 \cdot 3 = 18$$

$$\rightsquigarrow P(A) = 18/42 = 3/7$$

Second way: pull two at once

$$|S| = \binom{7}{2} = 7 \cdot 6/2 = 21$$

$$|A| = \binom{4}{2} + \binom{3}{2} = 6 + 3 = 9$$

$$\rightsquigarrow P(A) = 9/21 = 3/7$$

MORE EXAMPLES

4. In poker, what is the probability of dealing a 4-of-a-kind?

$S = \{\text{poker hands}\} = \binom{52}{5} = 2{,}598{,}960$

$A = \{\text{4-of-a-kind hands}\}$

What is $|A|$?

Pick a kind: 4

Pick 4 of that kind: 1

Pick a 5th card: 48

$\rightsquigarrow P(A) = {}^{13 \cdot 48}/_{2{,}598{,}960} \approx .00024 \approx {}^{1}/_{4000}$

What about a full house?

S = same

What is $|A|$?

Pick 1st kind (13), 2nd kind (12),

3 of 1st kind $\binom{4}{3}$, 2 of 2nd $\binom{4}{2}$

$\rightsquigarrow P(A) = {}^{3744}/_{2{,}598{,}960} \approx .0014 \approx {}^{1}/_{700}$

SOME PROBABILITY RULES

THEOREM: Let S be the sample space of some experiment. Let A and B be events.

(i) $0 \leq P(A) \leq 1$

$P(\emptyset) = 0, \quad P(S) = 1$

(ii) $P(A^c) = 1 - P(A)$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

More generally:

$$P(A_1 \cup \cdots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots$$

Can rephrase all counting rules as probability rules.

APPLYING PROBABILITY RULES

EXAMPLE: A number from 1 to 100 is chosen at random. What is the probability it is...

a) divisible by 2, 3, or 5?
b) divisible by 2 and 3, but not 5?
c) divisible by 3 but not 2 or 5?
d) divisible by at most two of 2, 3, and 5?

$|S| = 100$

$A_k = \{1 \le n \le 100 : n \text{ is divisible by } k\}$

$P(A_k) = \lfloor \frac{100}{k} \rfloor / 100$

$A_j \cap A_k = A_{\mathrm{lcm}(j,k)}$ so if $\gcd(j,k)=1$ then $A_j \cap A_k = A_{jk}$

a) $P(A_2 \cup A_3 \cup A_5) = 74/100$

b) $P((A_2 \cap A_3) \setminus A_5) = 13/100$

c) $P(A_3 \setminus (A_2 \cup A_5)) = 12/100$

d) $P((A_2 \cap A_3 \cap A_5)^c) = 97/100$

MUTUAL EXCLUSIVITY

Two events A and B are mutually exclusive if $A \cap B = \emptyset$

Events $A_1, \dots, A_n$ are pairwise mutually exclusive if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

A special case of the last theorem:

If $A_1, \dots, A_n$ are pairwise mutually exclusive events, then

$$P(A_1 \cup \dots \cup A_n) = P(A_1) + \dots + P(A_n) \quad \text{(addition rule)}$$

EXAMPLE: A number from 1 to 100 is chosen at random. What is the probability that the number is divisible by 7 or 30?

$$A_7 \cap A_{30} = A_{210} = \emptyset \rightsquigarrow P(A_7 \cup A_{30}) = P(A_7) \cup P(A_{30}) = 14/100 + 3/100 = 17/100$$

APPLYING PROBABILITY RULES

1. What is the probability that a length 10 bit string (chosen at random) has at least one zero? at least two zeros?

2. What is the probability that a poker hand (dealt at random) is a flush? a straight? royal flush?

Note: A, 2, 3, 4, 5 and 10, J, Q, K, A are both straights.

Section 7.4
Probability Theory

DEFINITIONS

An experiment is a procedure that yields one of a given set of outcomes.

The sample space of the experiment is the set of possible outcomes.

$$S = \text{finite set}$$


Blaise Pascal


Pierre Laplace

An event is a subset of the sample space:

$$A \subseteq S$$

The probability of an event A, assuming each outcome of the experiment is equally likely, is:

$$P(A) = |A| / |S|$$

PROBABILITY FUNCTIONS

Say we do an experiment with outcomes $s_1, \ldots, s_n$. It might be that the s_i are not equally likely. For instance, consider an unfair die:

$$P(1) = 1/3$$
$$P(2) = P(3) = 1/12$$
$$P(4) = P(5) = P(6) = 1/6$$

What is the probability of rolling an even number?

$$P(2) + P(4) + P(6) = 1/12 + 1/6 + 1/6 = 5/12$$

Odd?

$$1 - 5/12 = 7/12$$

A 4, 5, or 6?

$$P(4) + P(5) + P(6) = 1/6 + 1/6 + 1/6 = 1/2$$

PROBABILITY FUNCTIONS

For an experiment with outcomes $S = \{s_1, \dots, s_n\}$, a probability function is a function

$$P : S \to \mathbb{R}$$

with (i) $0 \le P(s_i) \le 1$ for all i. (the ≤ 1 is redundant)

(ii) $P(s_1) + \cdots + P(s_n) = 1$

If $A \subseteq S$ is an event, then

$$P(A) = \sum_{s_i \in A} P(s_i)$$

If each s_i is equally likely, then $P(s_i) = 1/|S|$

so $P(A) = \sum_{s_i \in A} \frac{1}{|S|} = |A|/|S|$, as before

Still true that:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) $P(A^c) = 1 - P(A)$

THE MONTY HALL PROBLEM

"Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the other doors, opens another door, say No. 3, which has a goat. He then says to you, 'Do you want to pick door No. 2?' Is it to your advantage to take the switch?"

THE MONTY HALL PROBLEM

Intuitive explanation: There are 3 equally likely cases:

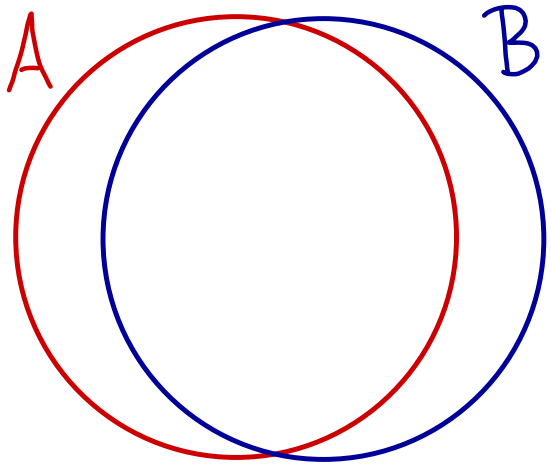
1. You initially picked a car $\rightsquigarrow$ switching gives a goat
2. You initially picked goat #1 $\rightsquigarrow$ switching gives a car
3. You initially picked goat #2 $\rightsquigarrow$ switching gives a car

So in $2/3$ of the cases, switching gives a car.

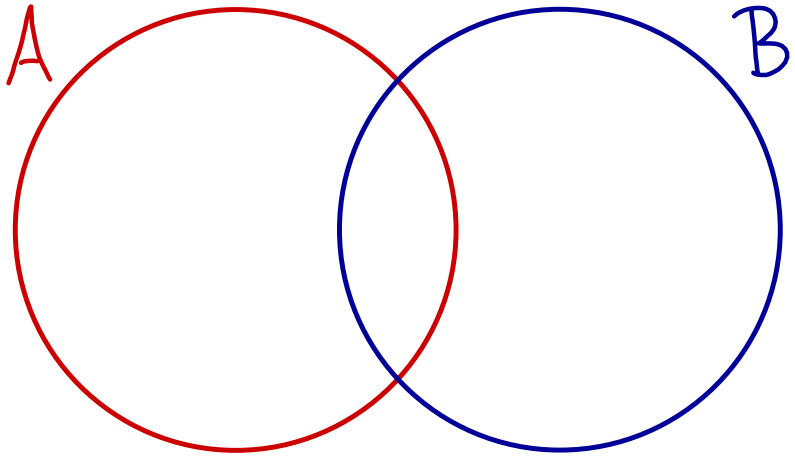
CONDITIONAL PROBABILITY

Say A and B are events and $P(A) > 0$. The conditional probability of B given A is:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$



$P(B|A)$ large



$P(B|A)$ small

$P(B)$ is same in both, but the knowledge of being in A makes a big difference.

CONDITIONAL PROBABILITY EXAMPLE

A coin is flipped twice. The first flip is heads. What is the probability that both flips are heads?

Intuition: $1/2$

Basic probability: $S = \{HH, HT\}$
$A = \{HH\}$
$\rightsquigarrow P(A) = 1/2$

Conditional probability: $P(HH \mid H*) = \dfrac{P(HH \cap H*)}{P(H*)} = \dfrac{1/4}{2/4}$

HH = both flips heads
$H*$ = first flip heads.

CONDITIONAL PROBABILITY EXAMPLE

I have two kids. One is a boy. What is the probability I have two boys?

S:

BB	BG
GB	GG

B = 2 boys

A = at least 1 boy

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{3/4} = 1/3$$

CONDITIONAL PROBABILITY EXAMPLES

1. An urn has 10 white, 5 yellow, and 10 black marbles. A marble is chosen at random. We are told it is not black. What is the probability it is yellow?

$$P(Y|B^c) = \frac{P(Y \cap B^c)}{P(B^c)} = \frac{5/25}{15/25} = 1/3$$

2. We deal bridge hands at random to N, S, E, W. Together, N and S have 8 spades. What is the probability that E has 3 spades?

We know E & W have 5 spades.

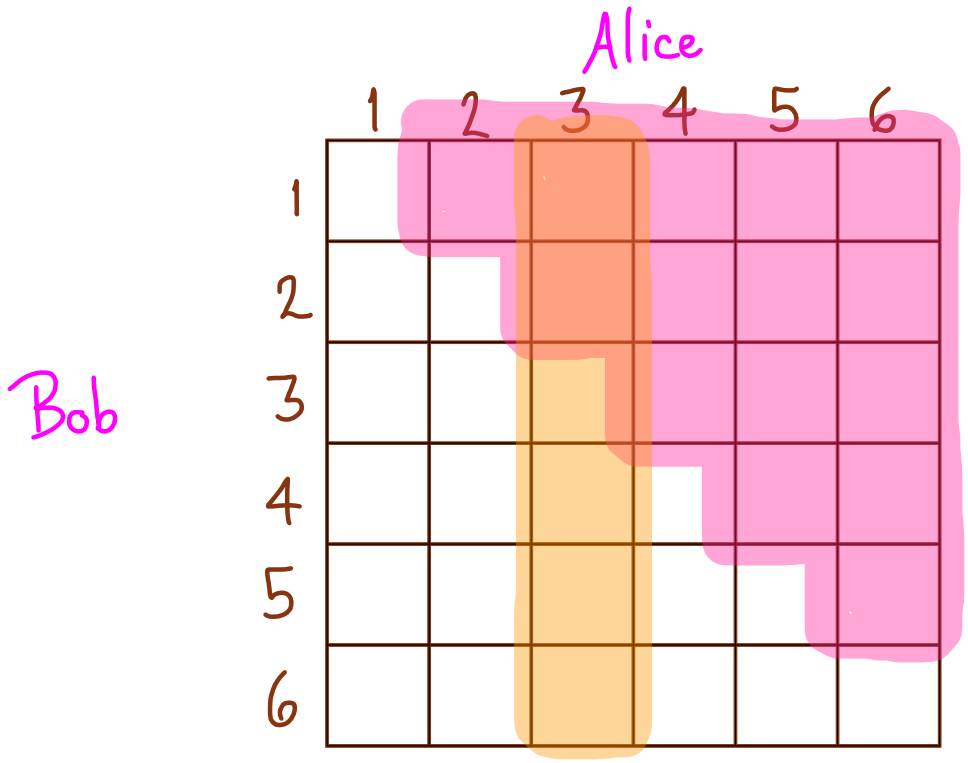
B = E has 3 spades, A = E & W have 5 spades

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\binom{5}{3}\binom{21}{10}/N}{\binom{26}{13}/N} = \frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}}$$

N = total # of ways of dealing hands.

CONDITIONAL PROBABILITY EXAMPLE

Alice and Bob each roll a die. We are told that Alice rolled a higher number. What is the probability that Alice rolled a 3?



B = Alice rolled 3

A = Alice > Bob

$$P(B|A) = P(B \cap A)/P(A) = \frac{2/36}{15/36} = 2/15.$$

INDEPENDENCE

Events A and B are independent if

$$P(B|A) = P(B)$$

Since $P(B) = \frac{P(B \cap A)}{P(A)}$ we can say A and B are independent if:

$$P(A \cap B) = P(A)P(B)$$

Examples. 1. We roll two die. A = first comes up 2

B = second comes up 3

$P(B|A) = \frac{1/36}{1/6} = 1/6 = P(B) \rightsquigarrow$ independent.

$P(A \cap B) = 1/36 = 1/6 \cdot 1/6 = P(A)P(B) \rightsquigarrow$ independent.

2. Two kids. B = 2 boys

A = at least one boy

$P(B|A) = 1/3 \neq 1/4 = P(B) \rightsquigarrow$ not independent.

INDEPENDENCE

Events A and B are independent if

$$P(B|A) = P(B)$$

Examples. 3. The Alice and Bob problem:

B = Alice rolled 3

A = Alice > Bob

$P(B|A) = 2/15$, $P(B) = 1/6$ $\rightsquigarrow$ not independent

4. Urn problem: 10 white, 5 yellow, 10 black.
Are Y and B^c independent?

$$P(Y|B^c) = 1/3 \neq 1/5 = P(Y)$$

$\rightsquigarrow$ not independent.

A CONDITIONAL PROBABILITY PROBLEM

We buy light bulbs from suppliers A and B.

30% of the bulbs come from A, 70% from B.

2% of the bulbs from A are defective

3% of the bulbs from B are defective.

What is the probability that a random bulb...

(i) is from A and defective?

(ii) is from B and not defective?

☆ (iii) is defective?

Given: $P(A) = 3/10$

$P(B) = 7/10$

$P(D|A) = 2/100$

$P(D|B) = 3/100$

(i) $P(A \cap D) = P(A)P(D|A) = \frac{3}{10} \cdot \frac{2}{100} = 3/500$

(ii) $P(B \cap D^c) = P(B) - P(B \cap D)$

$= P(B) - P(B)P(D|B)$

$= 7/10 - 7/10 \cdot 3/100 = 679/1000$

(iii) $P(D) = P(D \cap A) + P(D \cap B)$

$= P(A)P(D|A) + P(B)P(D|B) = 27/1000$

A CONDITIONAL PROBABILITY PROBLEM

We buy light bulbs from suppliers A and B.

30% of the bulbs come from A, 70% from B.

2% of the bulbs from A are defective

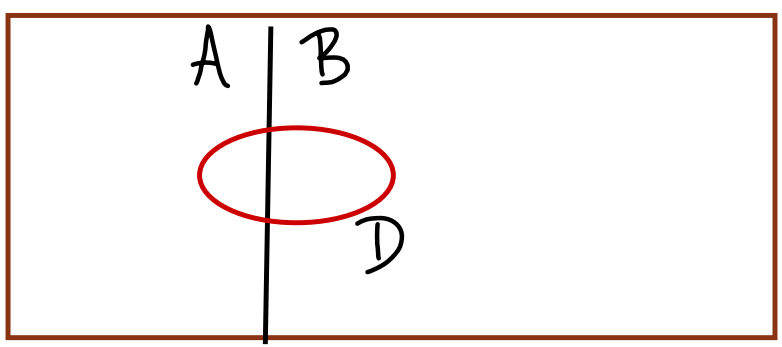
3% of the bulbs from B are defective.

What is the probability that a random bulb...

(i) is from A and defective?

(ii) is from B and not defective?

☆ (iii) is defective?



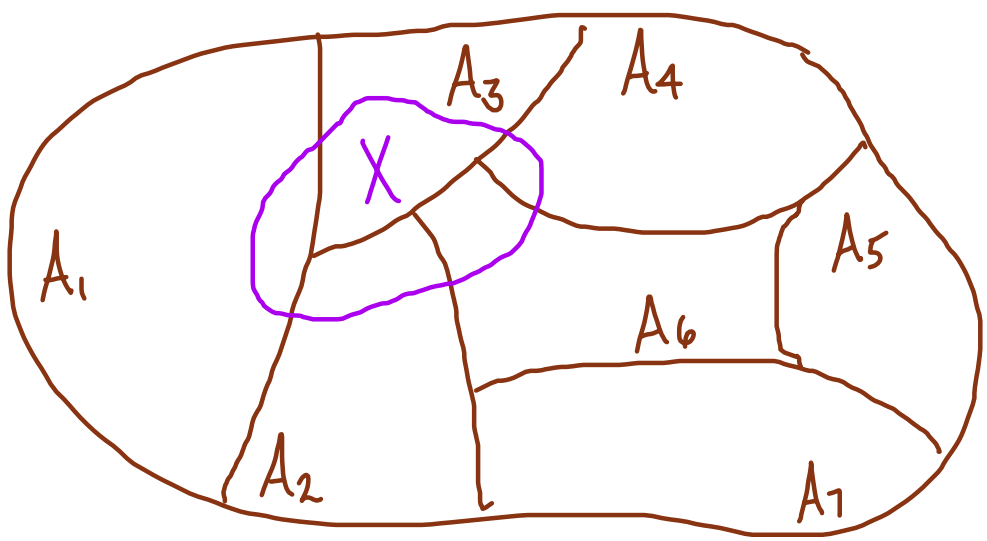
Reinterpret all questions in terms of areas.

LAW OF TOTAL PROBABILITY

Say that events $A_1, \ldots, A_n$ form a partition of the sample space S, that is, the A_i are mutually exclusive ($A_i \cap A_j = \emptyset$ for $i \neq j$) and $A_1 \cup \cdots \cup A_n = S$.

Let $X \subseteq S$ be any event. Then

$$P(X) = P(A_1)P(X|A_1) + \cdots + P(A_n)P(X|A_n)$$



BAYES' FORMULA

How is $P(A|B)$ related to $P(B|A)$?

THEOREM: $$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$

PROOF: $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{P(A)} \qquad \blacksquare$$

EXAMPLE. In the light bulb problem, say a randomly selected light bulb is defective. What is the probability it came from A?

$$P(A|D) = \frac{P(A)P(D|A)}{P(D)} = \frac{3/10 \cdot 2/100}{27/1000} = 2/9$$

BAYES' FORMULA

EXAMPLE. Coin A comes up heads 1/4 of the time.
Coin B comes up heads 3/4 of the time.
We choose a coin at random and flip it twice.
If we get two heads, what is the probability coin B was chosen?

$$P(B|HH) = \frac{P(B)P(HH|B)}{P(HH)} = \frac{1/2 \cdot 9/16}{P(HH)}$$

To find $P(HH)$, we use the law of total probability:

$$\begin{aligned} P(HH) &= P(HH|A)P(A) + P(HH|B)P(B) \\ &= 1/16 \cdot 1/2 + 9/16 \cdot 1/2 \\ &= 10/32 \end{aligned}$$

$$\rightsquigarrow P(B|HH) = \frac{9/32}{10/32} = 9/10$$

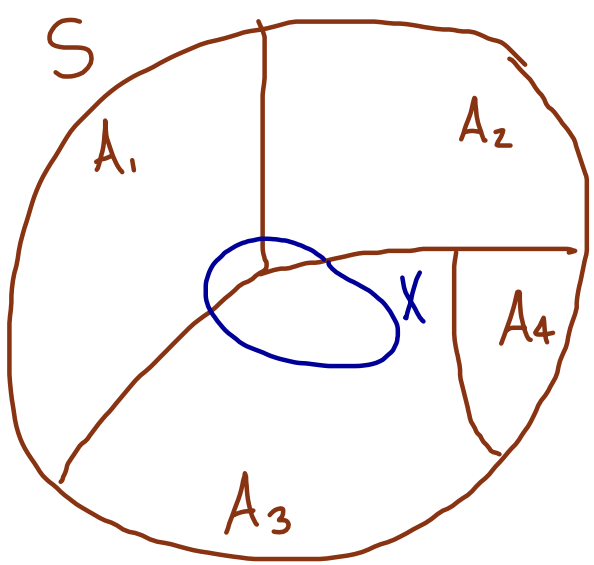
BAYES' FORMULA

Computing the denominator with the law of total probability

$A_1, \dots, A_n$ pairwise mutually exclusive events with $A_1 \cup \cdots \cup A_n = S$ and $P(A_i) > 0$ for all i. Let X be an event with $P(X) > 0$. Then, for each j, we have:

$$P(A_j | X) = \frac{P(A_j) P(X | A_j)}{P(X)}$$

where $P(X) = P(A_1)P(X|A_1) + \cdots + P(A_n)P(X|A_n)$



$P(A_3|X)$ big
$P(A_2|X)$ small
$P(A_4|X) = 0$.

EXAMPLE. Do a variant of the coin problem with 3 or more coins.

BAYES' FORMULA

PROBLEM. You have 3 cards. One is red on both sides, one is black on both sides, and one has a red side and a black side. You pick one card randomly and put it on the table. Its top side is red. What is the probability the other side is red?

$$P(RR|R) = \frac{P(R|RR)P(RR)}{P(R|RR)P(RR) + P(R|RB)P(RB) + P(R|BB)P(BB)}$$

$$= \frac{1 \cdot 1/3}{1 \cdot 1/3 + 1/2 \cdot 1/3 + 0 \cdot 1/3}$$

$$= \frac{1/3}{1/2} = 2/3$$

This problem is logically equivalent to the Monty Hall problem:
A, B, C = prize behind door A B C, You pick door A.
Door C opened, revealing goat. Want $P(B|\text{not } C)$.

BAYES' FORMULA

PROBLEM. There are 3 urns, A, B, and C that have 2, 4, and 8 red marbles and 8, 6, and 2 black marbles, respectively. A random card is picked from a deck. If the card is black we choose a marble from A, if it is a diamond we choose a marble from B, and otherwise choose a marble from C.

(a) What is the probability that a red marble gets drawn?

$2/5$

(b) If we know a red marble was drawn, what is the probability the card was hearts? diamonds?

$1/2$, $1/4$

Draw the picture!

Section 7.5
Repetitions

REPETITIONS

QUESTION: How many ways are there to put r identical marbles into n boxes, if you are allowed to put more than one marble per box?

First try 3 marbles into 10 boxes.

Case 1: All in same box $\binom{10}{1}$

Case 2: Two in one box, one in another $10 \cdot 9$

Case 3: All different boxes $\binom{10}{3} = 120$

Addition rule $\rightsquigarrow 120 + 90 + 10 = 220.$

What about 10 marbles in 3 boxes?

Lots of cases!

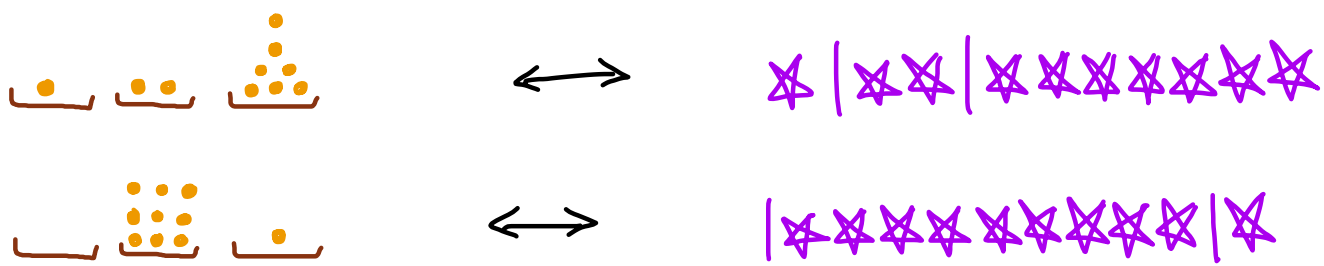
What to do?

Stars and Bars

Can answer the last question by looking at it the right way:

The number of ways of putting 10 marbles into 3 boxes is the same as:

the number of binary strings with 10 zeros, 2 ones
(or 10 stars, 2 bars)



How many such strings are there?

$\binom{12}{2} = 66$ (choose which of the 12 spots will be stars.)

REPETITIONS

QUESTION: How many ways are there to put r identical marbles into n boxes, if you are allowed to put more than one marble per box?

ANSWER: This is the same as the number of strings with r stars and $n-1$ bars:

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Repetitions, Permutations, and Combinations

How many ways to put r marbles in n boxes if...

	the marbles are indistinguishable	the marbles are distinguishable
at most one marble is allowed per box	$\binom{n}{r}$	$P(n,r)$
any number of marbles is allowed in a box	$\binom{n+r-1}{r}$	n^r

REPETITIONS

EXAMPLE: How many ways are there to choose 15 cans of soda from a cooler with (lots of) Coke, Dr. Pepper, Mtn Dew, RC cola, and Mr. Pibb?

Idea: Put 15 marbles in 5 boxes

$$\rightsquigarrow \binom{19}{15} = \binom{19}{4} = 3876$$

FURTHER: What if I insist on at least 3 Cokes and exactly one Mr. Pibb?

4 "marbles" already "used." One "box" already "full." $\rightsquigarrow \binom{14}{3} = 14 \cdot 13 \cdot 12 / 6 = 364$

REPETITIONS

EXAMPLE. In how many ways can we choose 4 nonnegative integers a, b, c, and d so that $a+b+c+d=100$?

100 marbles in 4 boxes: $\binom{103}{3} = 176{,}851$

What if a, b, c, and d are natural numbers?

4 marbles used $\rightsquigarrow \binom{99}{3} = 156{,}849$

REPETITIONS

EXAMPLE. How many ways are there to choose 4 integers $a, b, c,$ and d so that:

$$a+b+c+d=15$$
$$a \geq -3,\ b \geq 0,\ c \geq -2,\ d \geq -1\ ?$$

Set $a' = a+3$, $c' = c+2$, $d' = d+1$

$\rightsquigarrow$ $a' + b + c' + d = 21$ where $a', b, c', d \geq 0$.

$\rightsquigarrow$ 21 marbles, 4 boxes: $\binom{24}{3} = 2024$

Generalized Permutations

Example. How many ways are there to arrange the letters of SYZYGY?

Method 1: Choose 3 spots for Y: $\binom{6}{3}$
Then order the other letters: $3!$

120

Method 2: Arrange $SY_1ZY_2GY_3$ in $6!$ ways
Divide by $3!$

120.

Example. What about MISSISSIPPI?

34,650

Generalized Permutations

In general, say we have n objects that fall into k groups, with n_i objects in the ith group. Two objects in the same group are indistinguishable, but objects in different groups are distinguishable. In how many ways can we order the objects?

$$P(n; n_1, \ldots, n_k) = n! / n_1!\, n_2! \cdots n_k!$$

This is also the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in $(x_1 + x_2 + \cdots + x_k)^n$

Generalized Permutations

Example. Suppose there are 100 spots in the showroom of a car dealership. There are 15 (identical) sports cars, 25 compact cars, 30 station wagons, and 20 vans. In how many ways can the cars be parked?

$$P(100; 15, 25, 30, 20, 10) = 100!/15!\,25!\,30!\,20!\,10!$$

blanks

Section 7.6
Derangements

A Curious Probability

Question. A professor hands back exams randomly. What is the probability that no student gets their own exam?

Answer.

5 students ~ 36.8 %
10 students ~ 36.8 %
100 students ~ 36.8 %

DERANGEMENTS

A *derangement* of n objects that have some natural order is a rearrangement of the objects so that no object is in its correct position.

QUESTION. How many are there? Call the number D_n.

n	D_n	$P(D_n)$
1	0	0
2	1	$1/2$
3	2	$1/3$
4	9	$3/8$

What is the pattern?

A Formula For D_n

Let A_k be the permutations of n ordered objects with object k in the correct spot.

$$D_n = \left(\bigcup_{k=1}^{n} A_k\right)^c$$

$$\begin{aligned} D_4 &= 24 - |A_1 \cup A_2 \cup A_3 \cup A_4| \\ &= 24 - \sum |A_i| + \sum |A_i \cap A_j| - |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4| \\ &= 24 - \binom{4}{1} 3! + \binom{4}{2} 2! + \binom{4}{3} 1! + \binom{4}{4} 0! \\ &= 9 \end{aligned}$$

$$\begin{aligned} D_4 &= 4! - 4 \cdot 3! + \frac{4!}{2!2!} 2! - \frac{4!}{3!} + 1) \\ &= 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) \end{aligned}$$

Theorem. $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

D_N AND e

THEOREM. $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

Recall: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$$\rightsquigarrow e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$
$$\approx 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

So $D_n \approx n!/e$

$$\rightsquigarrow P(D_n) \approx \frac{n!/e}{n!} = 1/e \approx 0.368$$

For $n \geq 5$, this is correct to 3 decimal places.

DERANGEMENTS

PROBLEM. Fifteen people check coats at a party and at the end they are handed back randomly. How likely is it that...

(a) Tim gets his coat back?
(b) Jeremy gets his coat back?
(c) Jeremy and Tim get their coats back?
(d) Jeremy and Tim get their coats back but no one else does?
(e) The members of the Beatles get the right set of coats back (maybe not in the right order)?
(f) Everyone gets their coat back?
(g) Exactly one person gets their coat back?
(h) Nobody gets their own coat back?
(i) At least one person gets their coat back?

Section 7.7
The Binomial Theorem

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

⋮

Pascal's Triangle

Theorem. The kth entry in the nth row of Pascal's triangle is $\binom{n}{k}$ for $n \geq 0$ and $0 \leq k \leq n$.

Note: The top row is considered to be row 0, and the leftmost entry is entry 0.

Proof. We use induction on n.

Base case $n=0$ ✓

Assume true for $n=k-1$.

Entries in row $k-1$ look like: $\cdots \binom{n-1}{k-1} \quad \binom{n-1}{k} \cdots$

row n entry k

So the kth entry in row n is $\binom{n-1}{k-1} + \binom{n-1}{k}$.

Is this equal to $\binom{n}{k}$?

Yes – in choosing k objects, can either choose the nth object or not. Use the addition rule. ▨

Pascal's Triangle

① What is 11^n for $n = 0, 1, 2, \ldots$?

$$11^0 = 1$$
$$11^1 = 11$$
$$11^2 = 121$$
$$11^3 = 1{,}331$$
$$\vdots$$

② What is the sum of the entries in the nth row?

$$1 = 1$$
$$1 + 1 = 2$$
$$1 + 2 + 1 = 4$$
$$1 + 3 + 3 + 1 = 8$$
$$1 + 4 + 6 + 4 + 1 = 16$$
$$\vdots$$

The Binomial Theorem

Theorem. For any x and y and any natural number n, we have:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n} x^0 y^n$$

Proof. $(x+y)^n = (x+y)(x+y)\cdots(x+y)$

If we multiply out, we get an $x^{n-k}y^k$ term by choosing k of the x's. There are $\binom{n}{k}$ ways of doing this. $\blacksquare$

The Binomial Theorem

Problem. Expand $(2x^3+y)^5$ and simplify.

Problem. Expand $\left(x-\frac{1}{x}\right)^6$ and simplify.

Problem. Find the coefficient of x^{15} in $\left(x^2-\frac{x}{3}\right)^{11}$.

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

plug in...	to prove...	
$x=1,\ y=-1$	Inclusion-exclusion principle	
$x=10,\ y=1$	n^{th} row of P's $\Delta = 11^k$	← HW
$x=1,\ y=1$	n^{th} row sum of P's $\Delta = 2^n$	
$x=\sqrt{2},\ y=-1$	$\sqrt{2}$ is irrational	← HW

The Inclusion-Exclusion Principle

Theorem. $|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$

Proof: By the binomial theorem

$$0 = (1-1)^k = \binom{k}{0} - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k \binom{k}{k}$$

or $\binom{k}{1} - \binom{k}{2} + \cdots + (-1)^{k+1} \binom{k}{k} = 1$

Say an element of $\cup A_i$ is in k of the A_i. The left hand side counts the number of times that element is counted by the inclusion-exclusion formula. So every element is counted once. ▨

Row Sums in Pascal's Triangle

Theorem. The sum of the entries in the nth row of Pascal's triangle is 2^n.

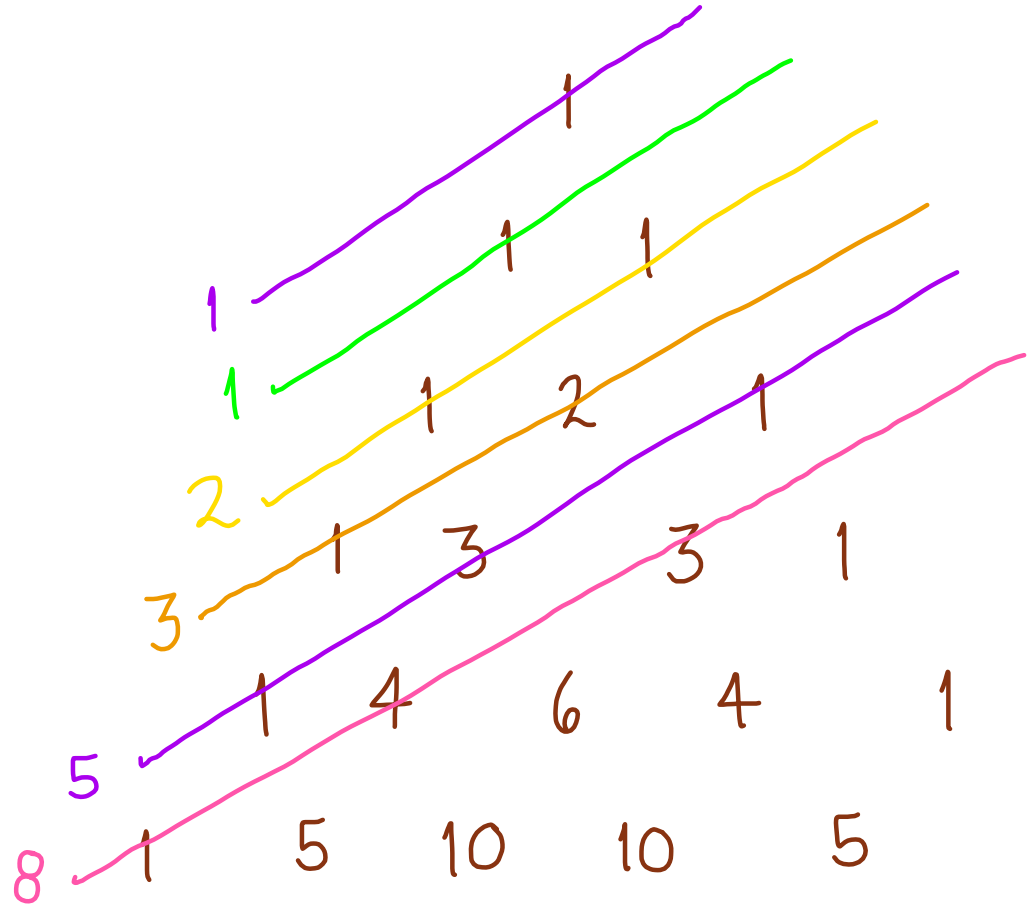
Proof. We have:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

But the $\binom{n}{k}$ are exactly the entries of the nth row of Pascal's triangle. ▨

Another Proof. We know the number of subsets of a set with n elements is 2^n. The sum shown above just counts all subsets according to their size. ▨

The Fibonacci Numbers In Pascal's Triangle

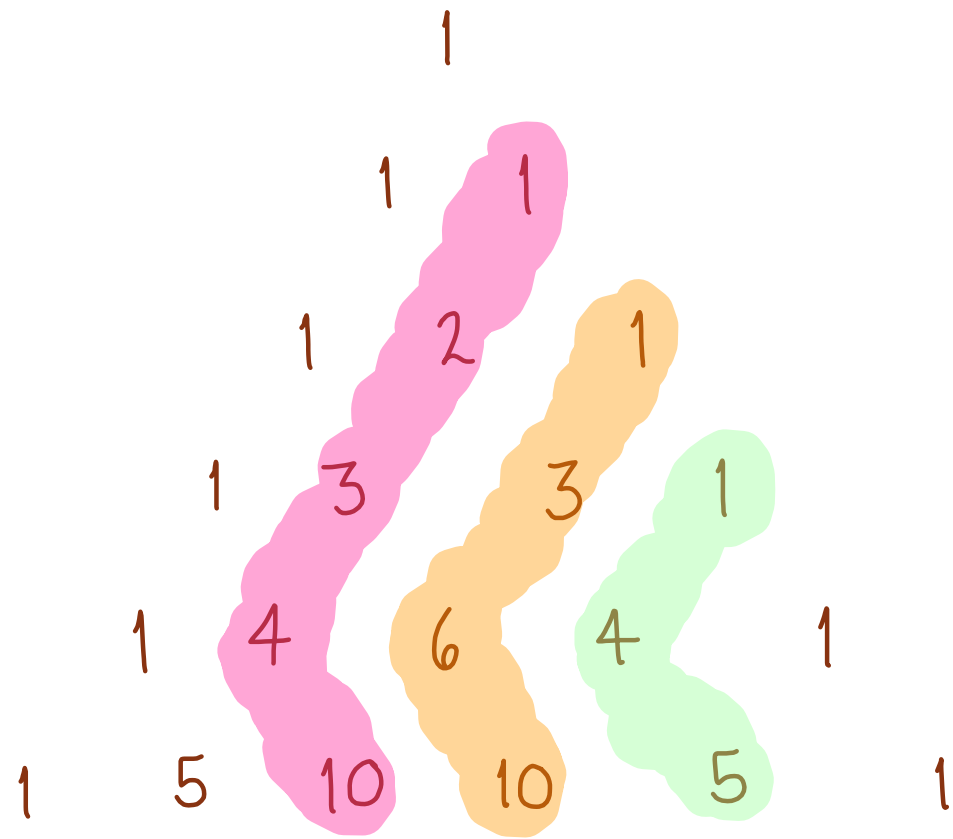


Theorem.

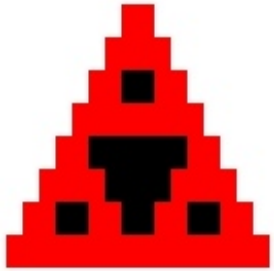
$$F_n = \begin{cases} \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{k}{k-1} & \text{if } n=2k \\ \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{k}{k} & \text{if } n=2k+1 \end{cases}$$

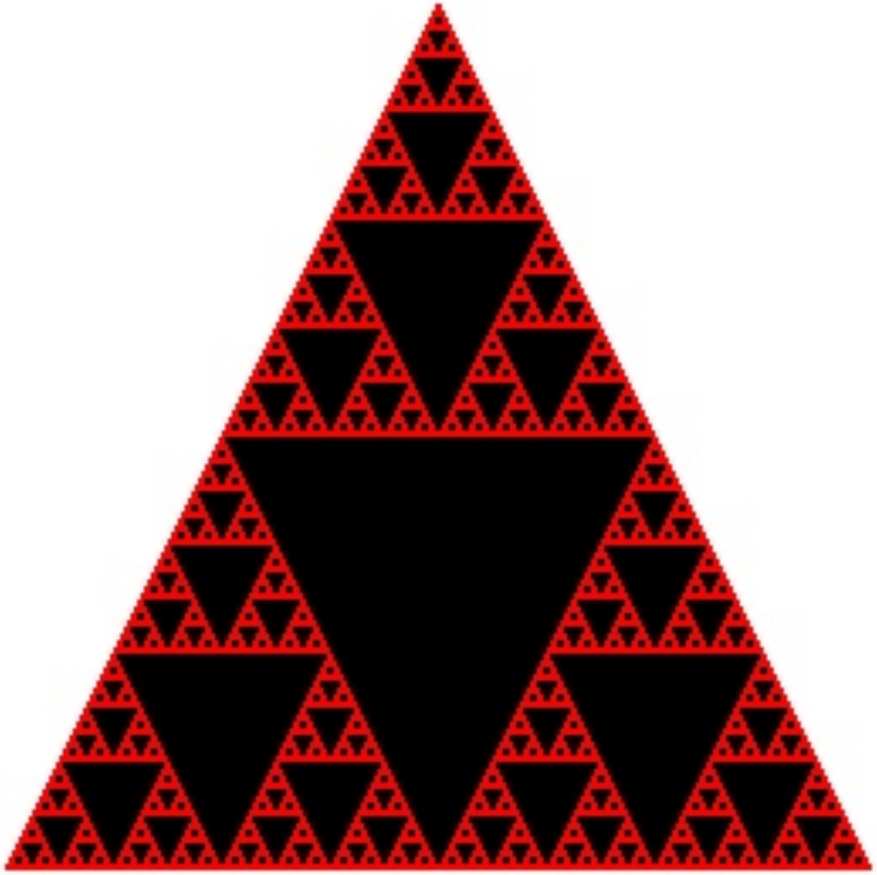
Proof. Use induction. Hint: each pink = purple + orange.

The Hockey Stick Theorem



Pascal's Triangle Mod 2





What about mod 3?

Chapter 9
Graphs

9.1 A Gentle Introduction

Four Problems

The Bridges of Konigsberg

Three House-Three Utility

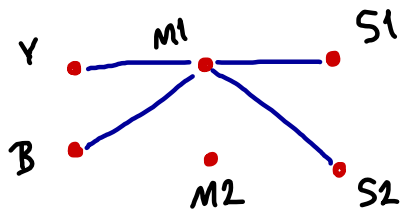
Four Color

Traveling Salesman

A Sample Problem

Among you, your buddy, two mothers, and two sisters, some people hug. There are no hugs between buddies, mothers, or sisters. The other 5 people tell you they all hugged different numbers of people. How many people did you hug?

Your buddy did not hug 4 people (otherwise noone hugged 0 people) $\rightsquigarrow$ may assume your mother hugged 4



So your buddy's mom hugged 0.
If your buddy hugged 3, nobody hugged 1.
If your buddy hugged 1, nobody hugged 3.
$\rightsquigarrow$ your buddy hugged 2 $\rightsquigarrow$ you hugged 2.

9.2 Definitions and Basic Properties

Graphs

A graph is a pair of sets V and E, where $V \neq \emptyset$ and each element of E is a pair of elements of V.

Write $G = G(V, E)$.

For us, graphs are finite, that is, $|V|$ is finite.

The elements of V and E are called vertices and edges.

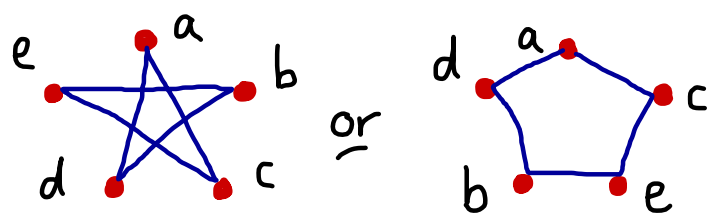
Example. V = Facebook users
E = Friendships

Graphs

We can represent graphs with pictures.

Example. Consider the graph $G(V,E)$ where

$$V = \{a,b,c,d,e\}$$
$$E = \{\{d,b\},\{a,c\},\{e,b\},\{e,c\},\{d,a\}\}$$



Can describe a graph with a picture instead of set notation.

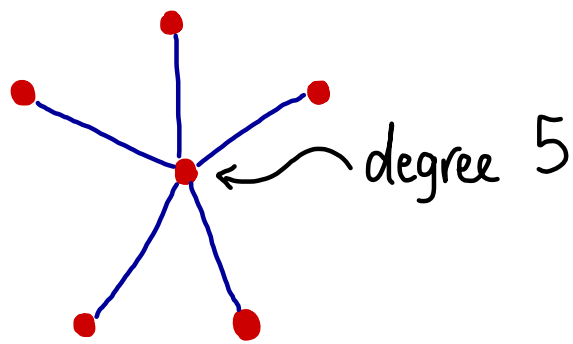
Could also write $E = \{db, ac, eb, ec, da\}$.

We say a is adjacent to c and d, and ac is incident to a and c.

DEGREES

The degree of a vertex v is the number of edges incident to v. Write $\deg v$.

If $\deg v = 0$, we say v is isolated.



Pseudographs

The following two phenomena are not allowed in a graph:

If we allow these, we get what is called a pseudograph.

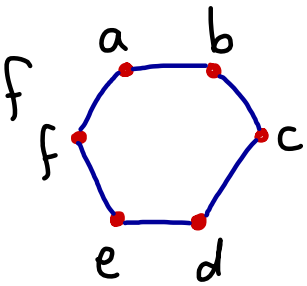
Pseudographs are harder to write down with set notation, so we usually describe them with a picture.

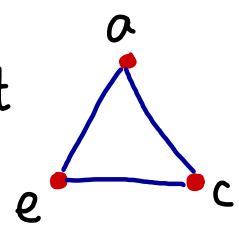
EXAMPLE. Vertices are web pages
Edges are links

Subgraphs

A subgraph of a graph $G(V,E)$ is a graph $G(V',E')$ where
$V' \subseteq V$ and
$E' \subseteq E$

Example. 

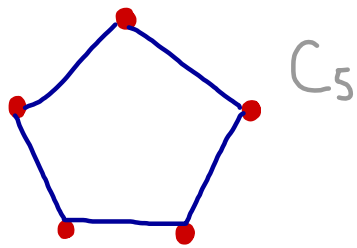
is a subgraph of 

but 

is not.

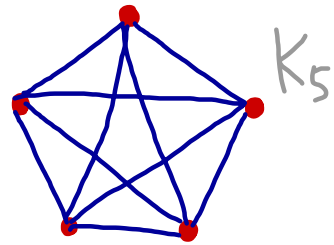
Also: Can delete any number of edges to get a subgraph.
Can delete any number of vertices (<u>and</u> all incident edges) to get a subgraph.

Three Special Families



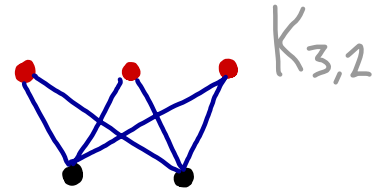
C_n

n-cycle



K_n

complete graph

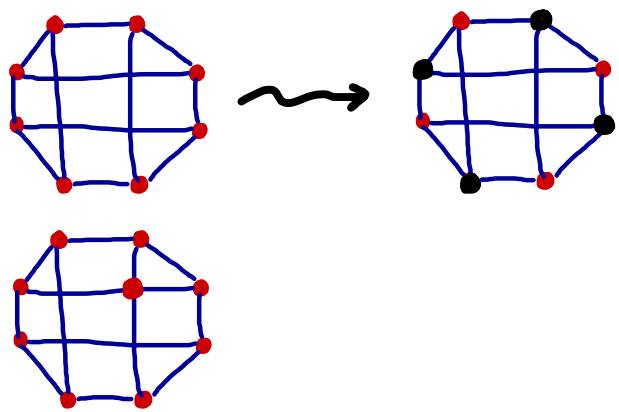


$K_{m,n}$

complete bipartite graph

Bipartite Graphs

A bipartite graph is one whose vertex set can be partitioned into two sets V_1 and V_2 so that each edge joins an element of V_1 to an element of V_2.



Fact. A bipartite graph contains no triangles. More generally, a bipartite graph contains no odd cycles.

The Handshaking Lemma

Proposition. The sum of the degrees of the vertices of a pseudograph is an even number. Specifically:

$$\sum_{v \in V} \deg v = 2|E|$$

Leonhard Euler

Handshaking Lemma. The number of odd degree vertices of a pseudograph is even.

Proof. $\displaystyle\sum_{v \in V} \deg v = \sum_{v \text{ even}} \deg v + \sum_{v \text{ odd}} \deg v$

Revisit the hugging problem.

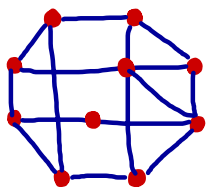
The Handshaking Lemma

Problem. A graph has 50 edges, 4 vertices of degree 2, 6 of degree 5, 8 of degree 4, all other vertices have degree 6. How many vertices does the graph have?

Problem. Out of 24 curling players, 78 pairs have played on the same team. Show that one has played on the same team as 7 others. Show that one has played on the same team with no more than 6 others.

Degree Sequence

Say $d_1, \ldots, d_n$ are the degrees of the vertices of a pseudograph, where $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $d_1, \ldots, d_n$ is the degree sequence of the pseudograph.



$\rightsquigarrow$ 5, 4, 3, 3, 3, 3, 3, 3, 3, 2

9.3 Graph Isomorphism

Graph Isomorphism

Two graphs $G(V,E)$ and $G(V',E')$ are isomorphic if there is a bijection

$$V \longrightarrow V'$$

that preserves adjacency and nonadjacency.

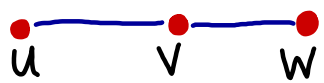
In other words, two graphs are isomorphic if there is a change of labels taking one to the other.

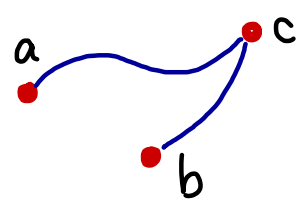
Example. $V = \{u, v, w\}$ $\quad$ $V' = \{a, b, c\}$ $\quad$ $V \longrightarrow V'$

$E = \{uv, vw\}$ $\quad$ $E' = \{ac, cb\}$ $\quad$ $u \longmapsto a$

$w \longmapsto b$

$v \longmapsto c$

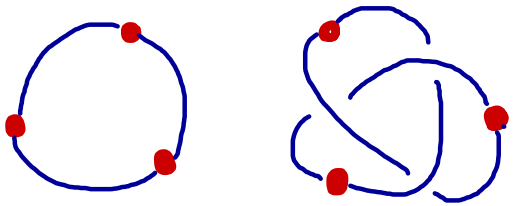




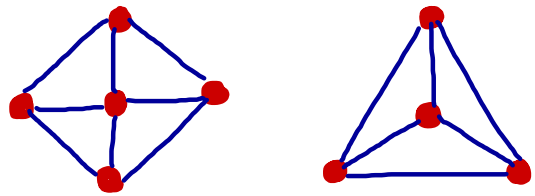
Graph Isomorphism

Which of the following pairs are isomorphic?

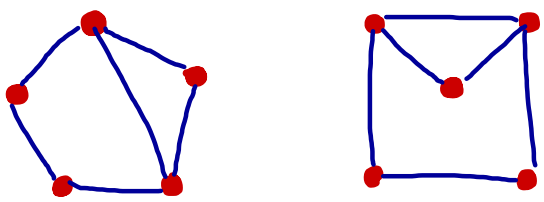
a)



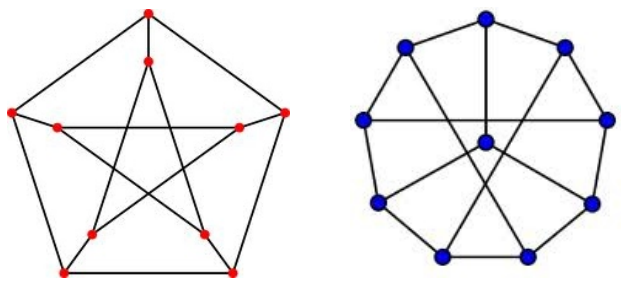
b)



c)



d)

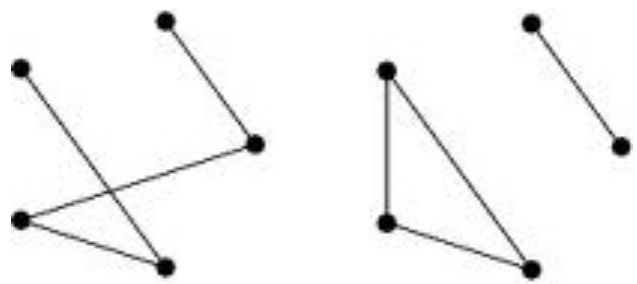


Invariants of Graphs

We can use the following "fingerprints" of graphs in order to tell if two graphs are different:

(i) Number of vertices
(ii) Number of edges
(iii) Degree sequence
etc.

It is possible for two graphs to have the same degree sequence and be nonisomorphic:

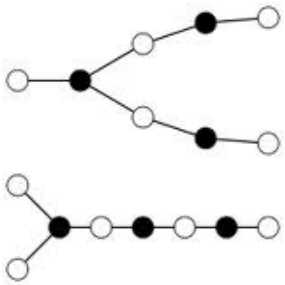


$\{2, 2, 2, 1, 1\}$ $\{2, 2, 2, 1, 1\}$

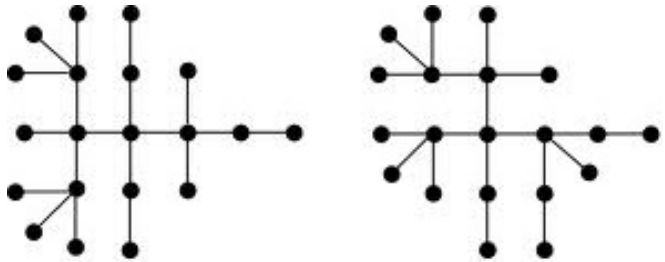
Examples

Which of the following graphs are isomorphic?

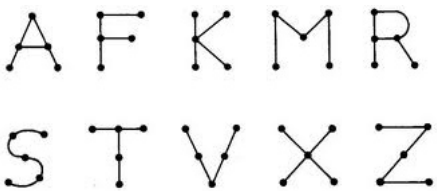
a)



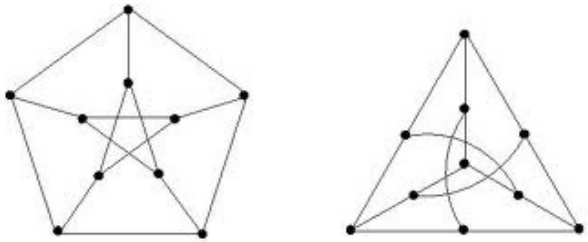
b)



c)



d)



Chapter 10
Paths and Circuits

10.1 Eulerian Circuits

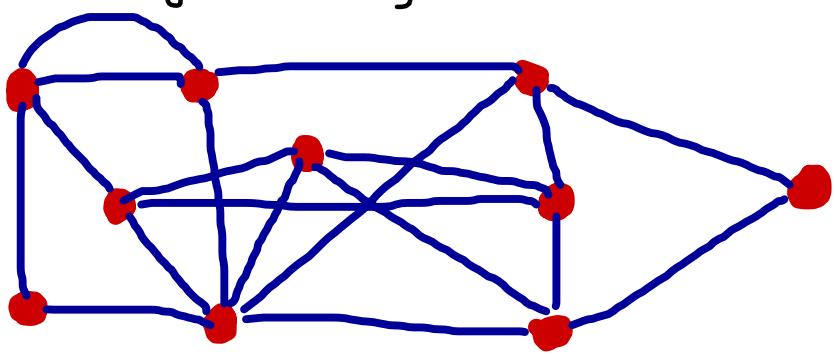
Eulerian Circuits

A walk is an alternating sequence of vertices and edges

$$v_1 e_1 v_2 e_2 \cdots v_n e_n v_1$$

where $e_i = v_i v_{i+1}$ and the e_i are distinct.

An Eulerian circuit in a pseudograph is a walk that crosses each edge exactly once and ends where it started.



Find one!

If a pseudograph G admits an Eulerian circuit, we say that G is Eulerian.

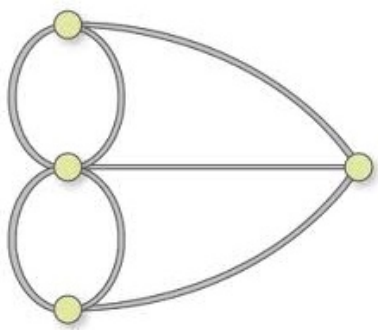
Euler is pronounced Oiler.

The Königsberg Bridge Problem

The Bridges of Konigsberg

Is it possible to take a walk, cross each bridge exactly once, and return to where you started?

Or: Is the following pseudograph Eulerian?



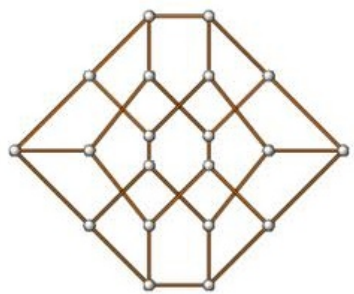
Answer: No! Each vertex needs to have even degree.

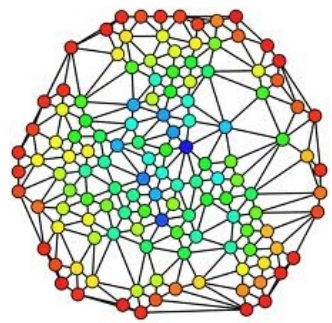
Connectivity

We just argued that Eulerian graphs have no vertices of odd degree.

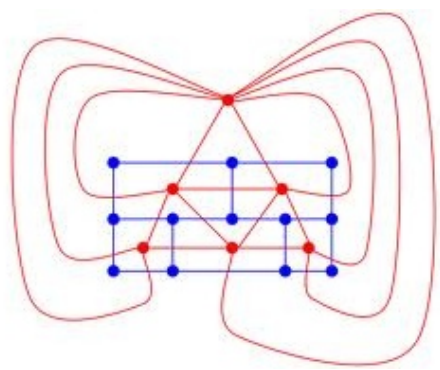
What else? Eulerian graphs must also be connected.

A pseudograph is connected if there is a walk between any two vertices.





connected



not connected

Eulerian Pseudographs

Theorem. A pseudograph is Eulerian if and only if it is connected and every vertex has even degree.

Proof. We already know Eulerian pseudographs are connected with no odd vertices.

Say G is connected and has no odd vertices.
Start at any vertex, walk across an adjacent edge.
Because every vertex is even, can keep walking. When we run out of options for continuing, we must be back at starting point.

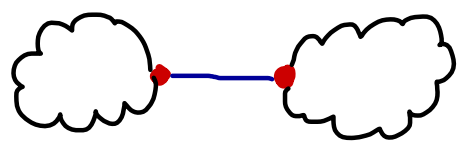
If we haven't used all edges, then repeat the process.

At the end, since G is connected, we can amalgamate these walks into one walk.

Eulerian Pseudographs

Theorem. A pseudograph is Eulerian if and only if it is connected and every vertex has even degree.

Second Proof (Fleury's Algorithm). Pick a starting vertex.
While the pseudograph has at least one edge:
traverse any edge that is not a bridge



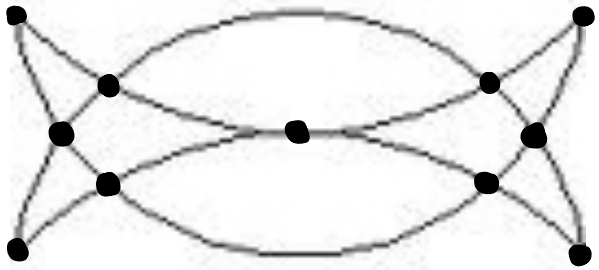
and delete that edge.

Prove that this always results in an Eulerian circuit.

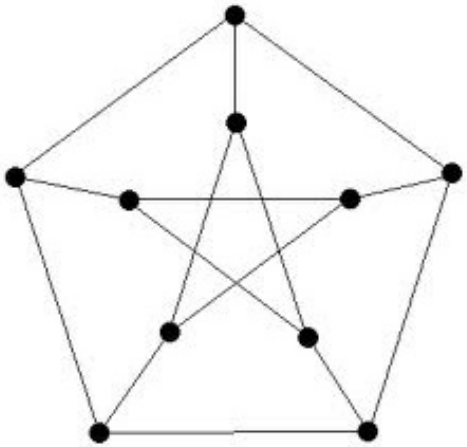
Eulerian Pseudographs

For each pseudograph, find an Eulerian circuit if it exists.

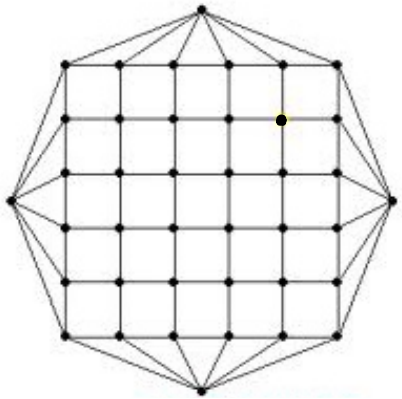
(i)



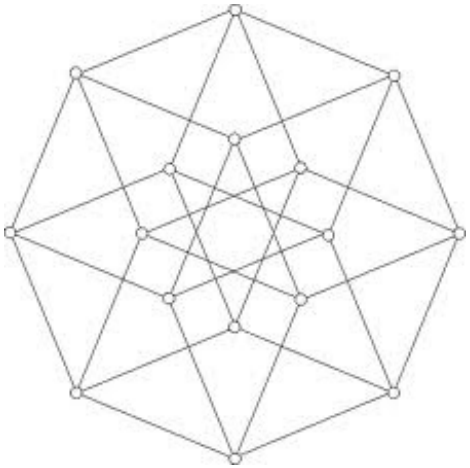
(ii)



(iii)



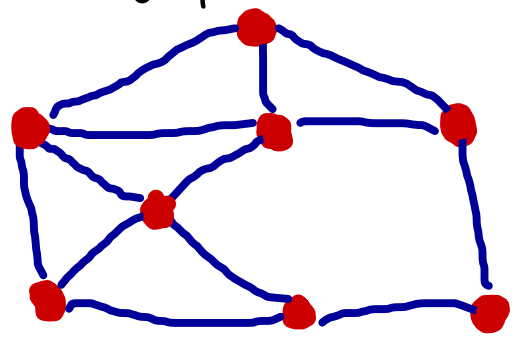
(iv)



10.2 Hamiltonian Cycles

Hamiltonian Cycles

A Hamiltonian cycle in a pseudograph is a walk that visits each vertex exactly once:



Find one!

If a pseudograph has a Hamiltonian cycle, we say the pseudograph is Hamiltonian.

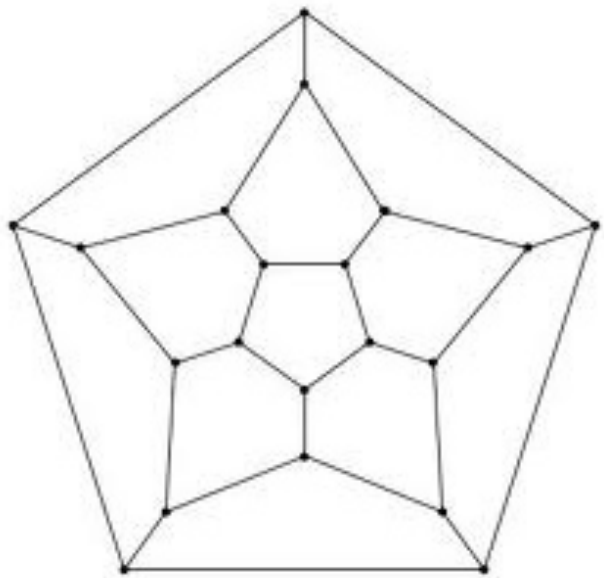
Euler: each edge once
Hamilton: each vertex once

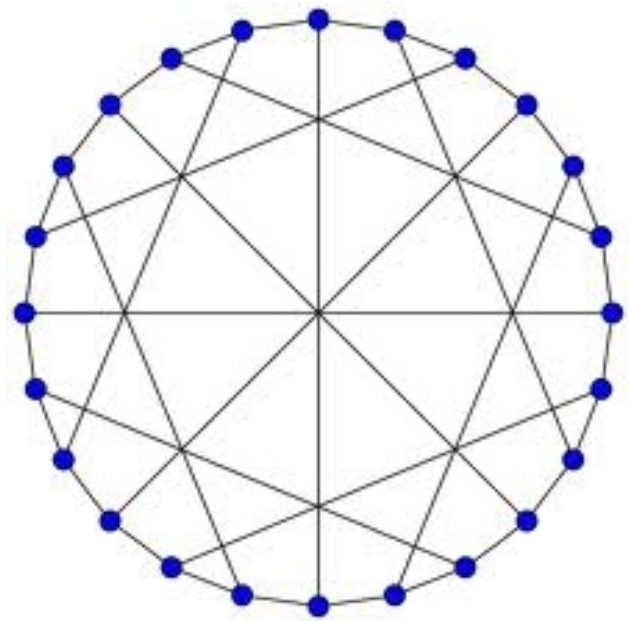
Sir William Rowan Hamilton

Note: A Hamiltonian cycle is isomorphic to an n-cycle.

Hamiltonian Cycles

Show that the following graphs are Hamiltonian.





In other words, find a Hamiltonian cycle in each.

Hamiltonian Graphs

We saw that it is easy to tell if a graph is Eulerian or not.
To prove a graph is Hamiltonian, just find a Hamiltonian cycle.
But there is no easy method for showing a graph is not Hamiltonian.

You could check all paths of length $|V|$. Takes too long!

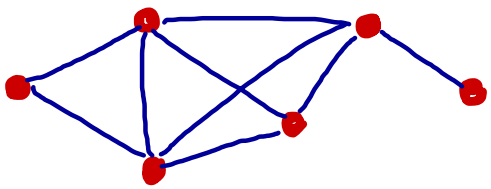
Better to use some basic facts:

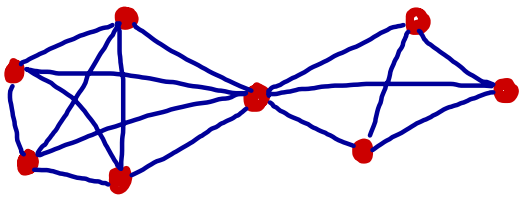
Let $\mathcal{H}$ be a Hamiltonian cycle in a pseudograph G

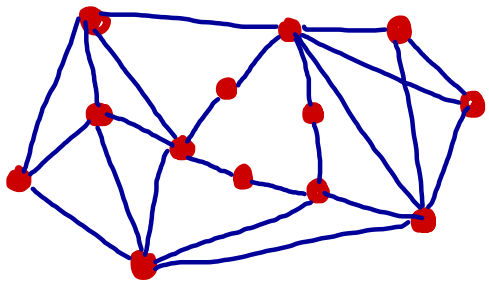
1. Every vertex of G has exactly two edges of $\mathcal{H}$ passing through it.
2. The only cycle contained in $\mathcal{H}$ is $\mathcal{H}$.

Hamiltonian Graphs

Prove that the following graphs are not Hamiltonian.

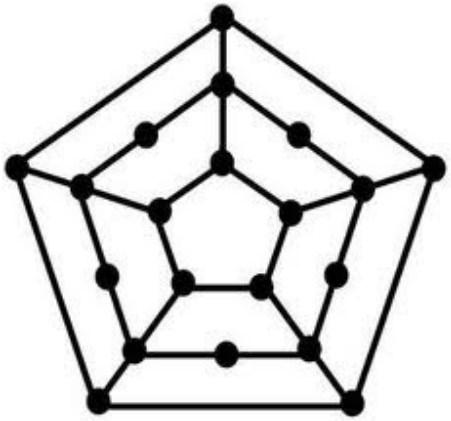


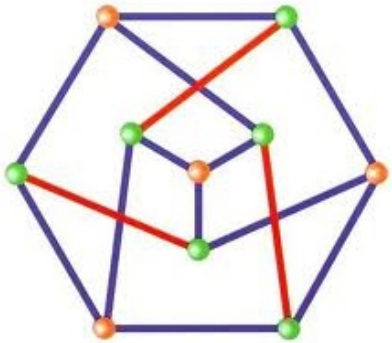


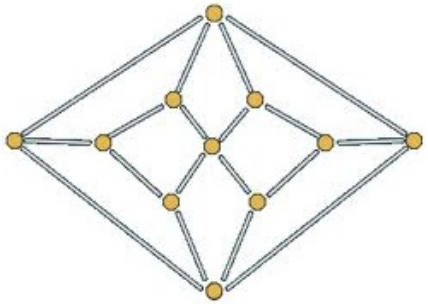


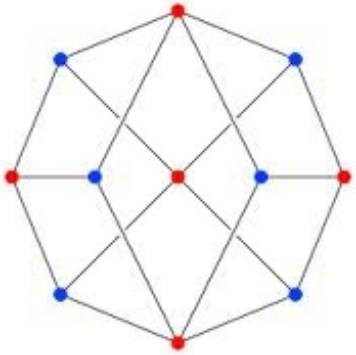
Hamiltonian Graphs

Which of the following graphs are Hamiltonian?

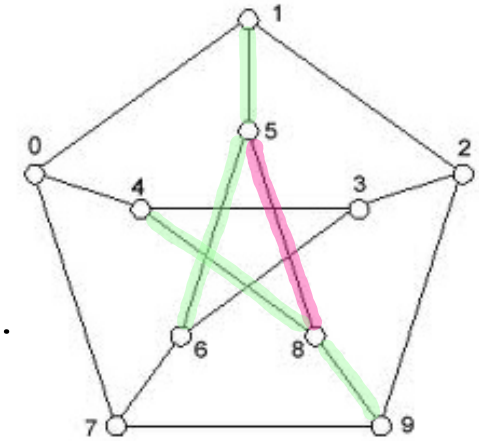








The Petersen Graph



Proposition. The Petersen graph is not Hamiltonian.

Proof. Say $\mathcal{H}$ is a Hamiltonian cycle. $\mathcal{H}$ must contain an edge connecting the inner and outer pentagons, say 15. From 5, need either 56 or 58, say 56. But then 58 is not in $\mathcal{H}$, so 89 and 84 are in $\mathcal{H}$. See

Now there are two cases for the other edge of $\mathcal{H}$ at 4:

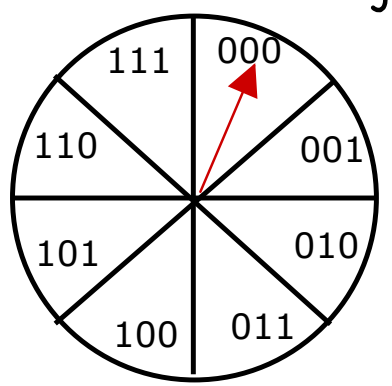
Case 1. 40 in $\mathcal{H}$.

In this case, 43 is not in $\mathcal{H}$ $\rightsquigarrow$ 32 & 36 in $\mathcal{H}$ $\rightsquigarrow$ 76 not in $\mathcal{H}$ $\rightsquigarrow$ 70 & 79 in $\mathcal{H}$ $\rightsquigarrow$ The cycle 07984 in $\mathcal{H}$, contradicting Fact 2.

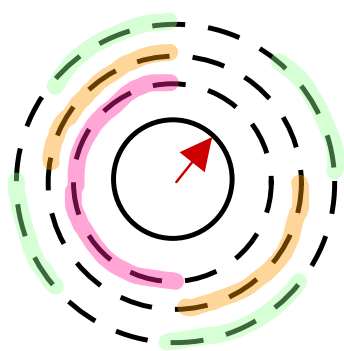
Case 2. 43 in $\mathcal{H}$. Similar. $\blacksquare$

Gray Codes

We can record the position of a rotating pointer with a bit string:



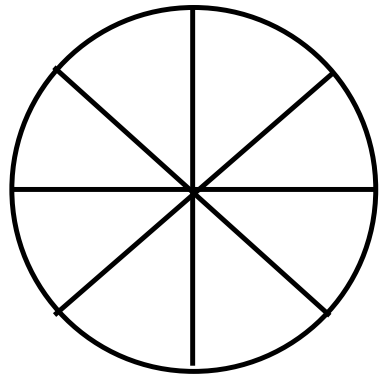
Can read the position of the arrow with 3 sets of contacts:



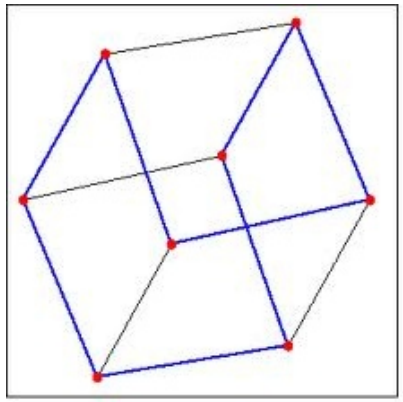
Problem: A small error could give 100 instead of 011
$\rightsquigarrow$ all 3 bits wrong!

Gray Codes

To fix this, want to number so that adjacent regions differ by one bit.



At first, not obvious how to do this.
But: such a numbering is just a Hamiltonian cycle in the n-cube.

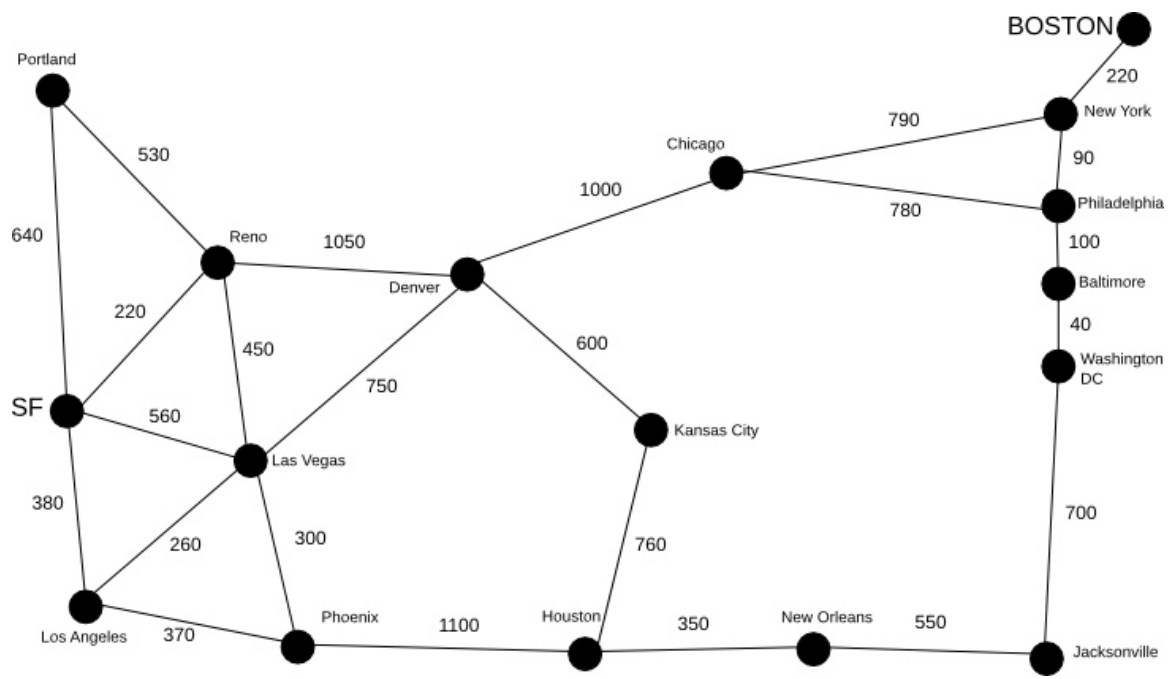


10.4 Shortest Path Algorithms

Weighted Graphs

A weighted graph is a graph $G(V,E)$ together with a function
$w: E \to [0,\infty)$

For $e \in E$, the number $w(e)$ is the weight of e.



Weighted Graphs

Graph	Vertices	Edges	Weights
communication	computers	fiberoptic cables	response time
air travel	airports	flights	flight times
car travel	street corners	streets	distances
Kevin Bacon	actors	common movies	1
stock market	stocks	transactions (directed edges)	cost
operations research	projects	dependencies (directed edges)	times

Distance Problems

Traveling Salesman Problem. Given a list of cities to visit, what is the minimum distance you need to travel?

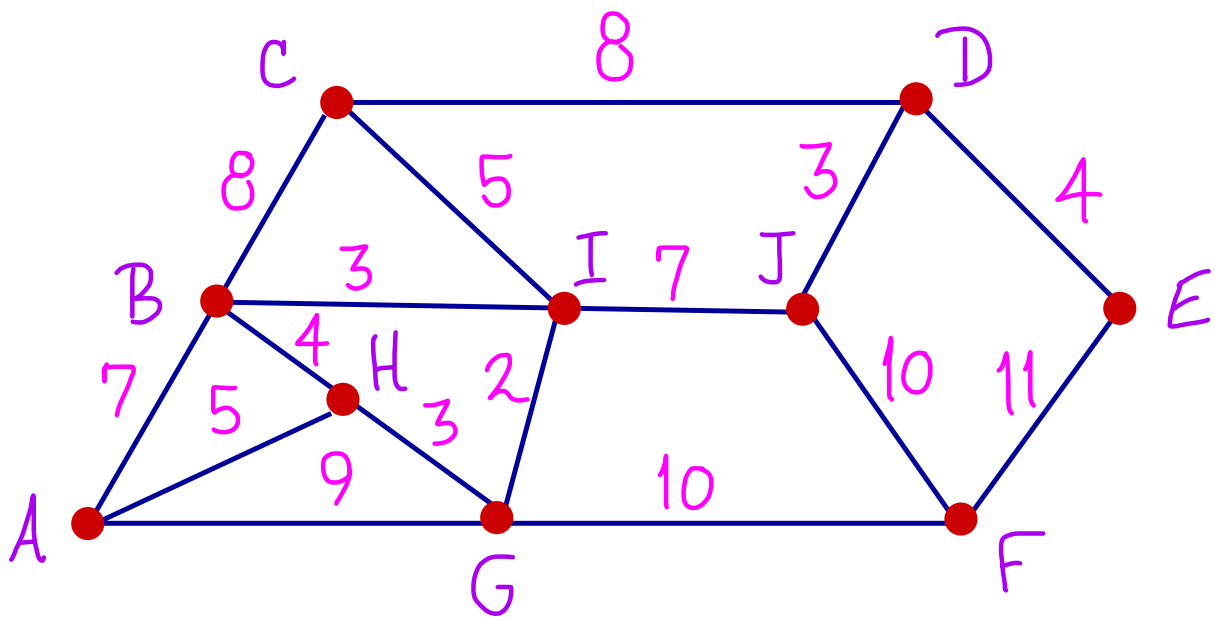
TSP is really a question about weighted graphs.

Easier Problem. Given two vertices in a weighted graph, what is their "distance."

The length of a walk is the sum of the weights of the edges traversed, and the distance between two vertices is the minimum length of a walk between them.

EXAMPLE

PROBLEM. Find the distance between A and E.



How to find the shortest path in general?

Dijkstra's Algorithm

To find the distances from a given vertex A in a weighted graph to all other vertices, do the following.

First, give A the permanent label 0, and give all other vertices the temporary label ∞.

Then repeat the following step:

Find the vertex v with the newest permanent label.

For each vertex v' adjacent to v with a temporary label, check if

$$\text{label of } v + w(vv') \leq \text{label of } v'$$

If so, change the temporary label of v'.

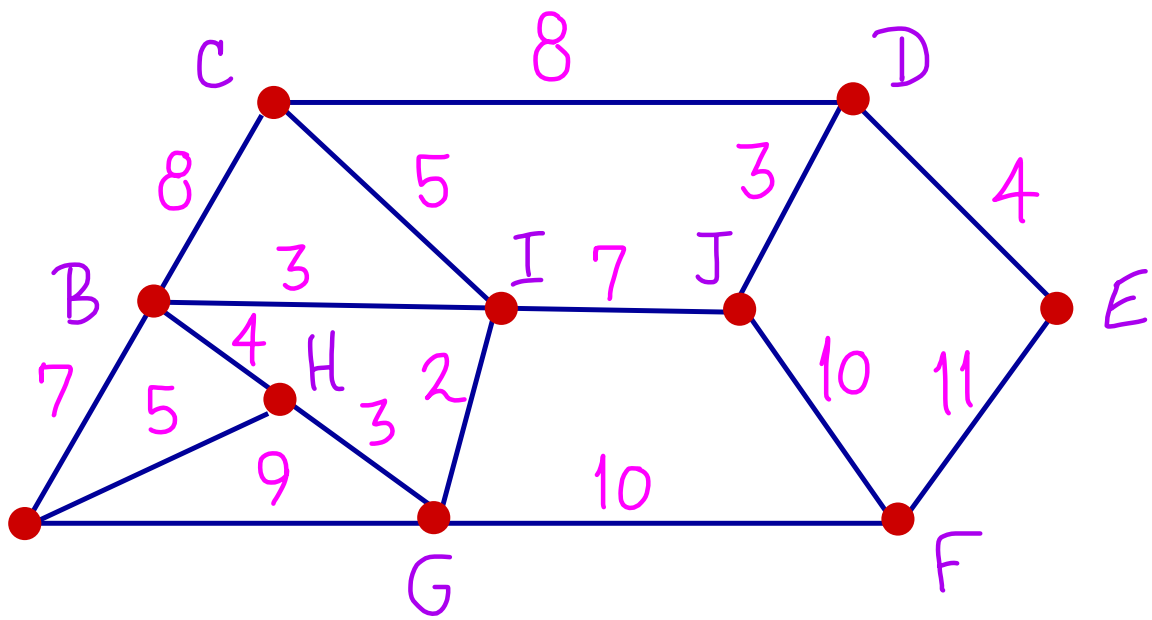
Make the smallest temporary label permanent.

Edsger Dijkstra

Permanent labels are the distances from A.

Dijkstra's Algorithm

Find the distance from A to each other vertex.



Dijkstra's Algorithm

Why does Dijkstra's algorithm work?

Use induction on the number of edges needed to walk from A to the other vertices.

Base case: 0 The only such vertex is A, whose permanent label and distance from A are both 0.

Inductive step. Suppose we need to cross at least one edge to get from A to v. If there is a path of length d from A to v, there is a path of length $d' < d$ from A to some vertex v' that is adjacent to v and is one "step" closer to A. By induction, the permanent label of v' is its distance from A. It then follows that v will get the correct permanent label. (Why?)

Dijkstra's Algorithm

What is the complexity of Dijkstra's algorithm, if size is measured in the number of vertices and cost is measured in terms of number of operations (= additions and comparisons)?

At k^{th} step, there are $n-k$ vertices without a permanent label.

$\rightsquigarrow$ at most $n-k$ additions, $n-k$ comparisons.

Then need $n-k-1$ comparisons to find the smallest temporary label.

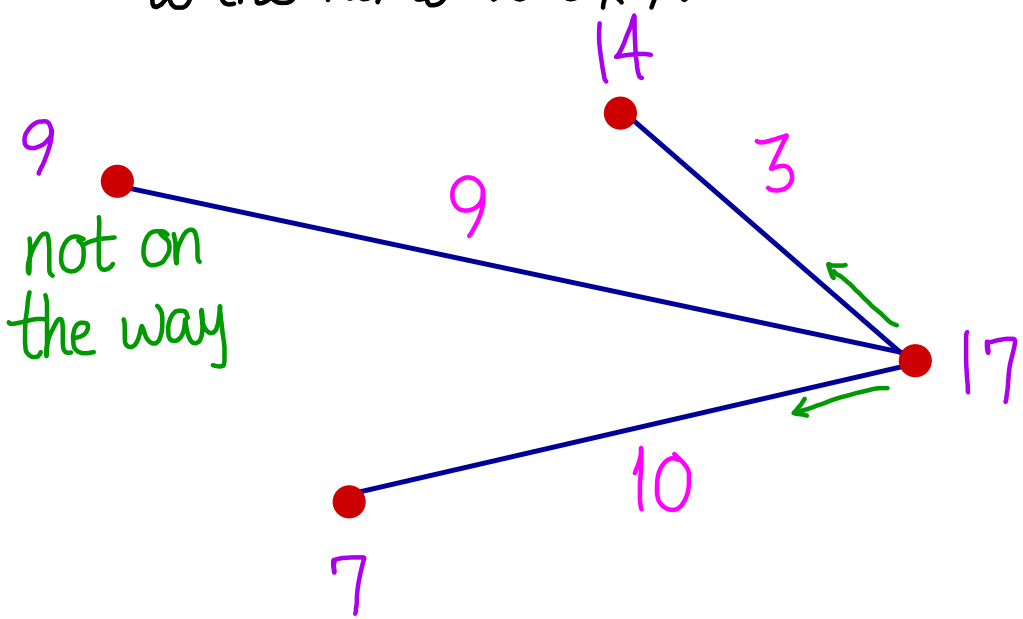
$$f(n) = \sum_{k=1}^{n-1} \left(2(n-k) + (n-k+1) \right)$$

$$= \frac{3}{2}n^2 - \frac{5}{2}n + 1 = O(n^2)$$

Dijkstra's Algorithm

What if we further want to find a walk between two vertices with the shortest length (not just the distance between the two vertices)?

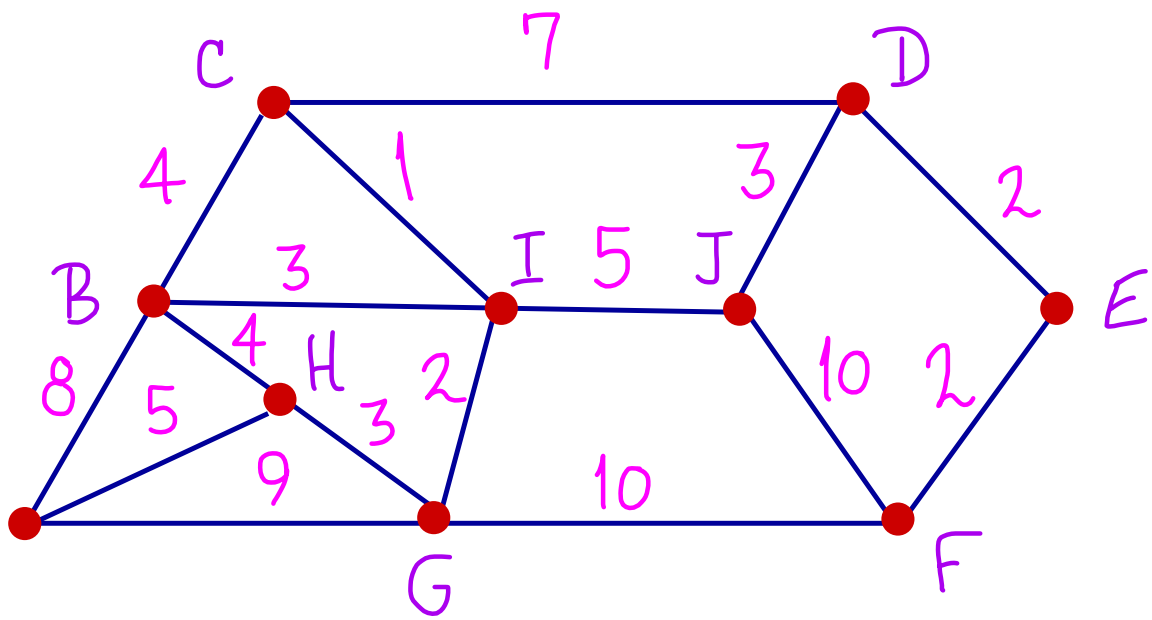
Idea: Every time we make a label permanent, draw a little arrow from that vertex to all other vertices that are "en route" to the home vertex A



Then, follow the arrows to find all shortest walks home.

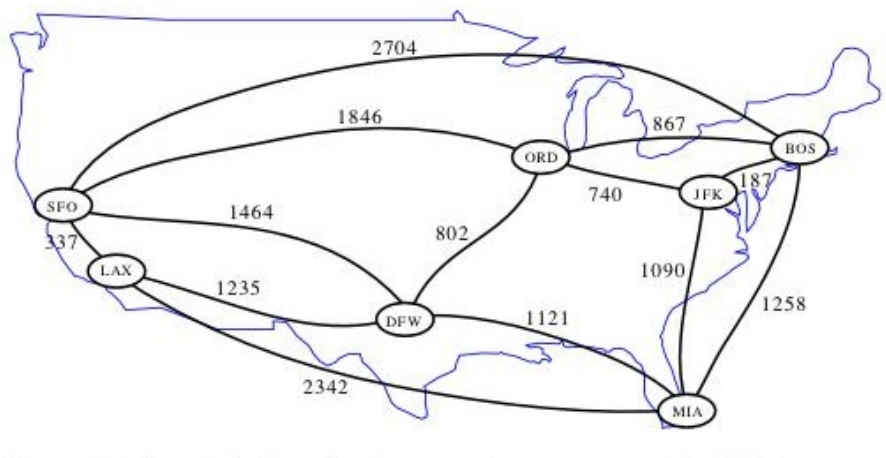
Dijkstra's Algorithm

Find all shortest paths from A to E.

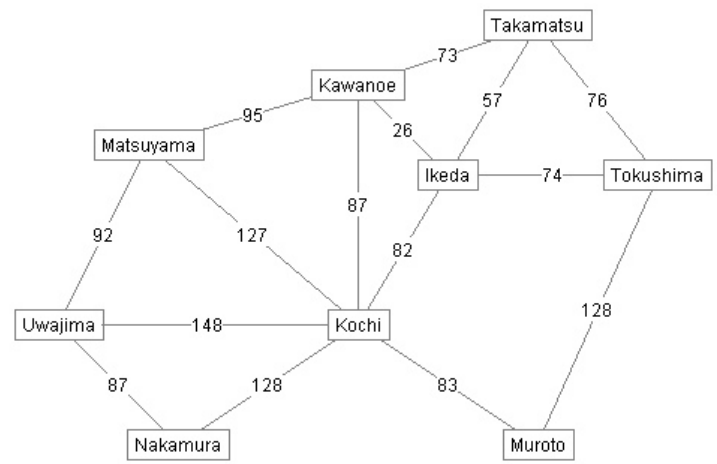


Dijkstra's Algorithm

Find the shortest paths...



from LAX to JFK



from Nakamura to Tokushima

Floyd-Warshall Algorithm

Idea: Number the vertices $v_1, \dots, v_n$.

Step k: Find the shortest path from v_i to v_j if you are only allowed to use $v_1, \dots, v_k$ as intermediate vertices (= pit stops).

Can write this info in a matrix M_k.

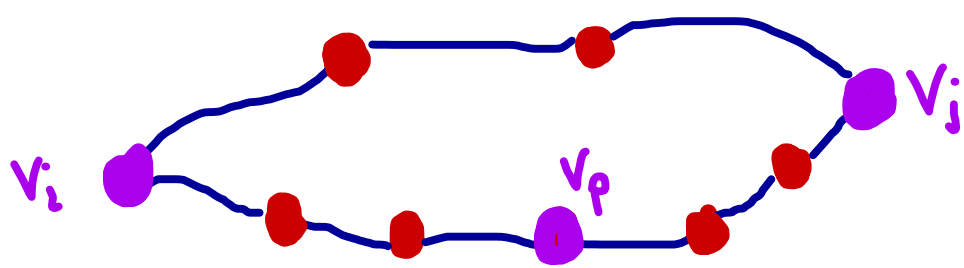
Write ∞ if there is no path.

Do this for $k = 0, \dots, n$. (At Step 0, no pit stops allowed.)

The ij-entry of M_n is the distance from v_i to v_j.

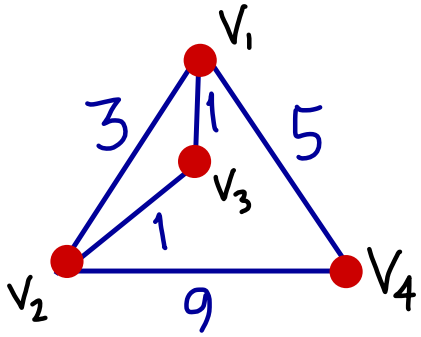
Key observation. For $k \geq 1$:

$$M_k(i,j) = \min_p \{ M_{k-1}(i,j),\ M_{k-1}(i,p) + M_{k-1}(p,j) \}$$



Floyd-Warshall Algorithm

Example.



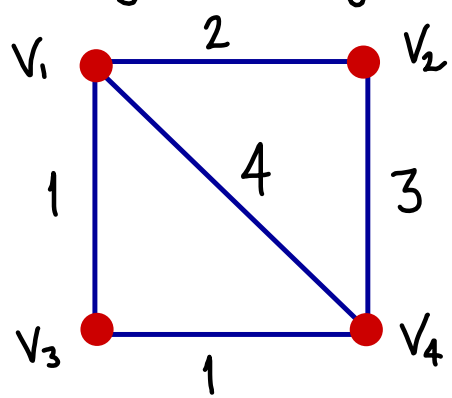
$$M_0 = \begin{pmatrix} 0 & 3 & 1 & 5 \\ & 0 & 1 & 9 \\ & & 0 & \infty \\ & & & 0 \end{pmatrix} \quad M_1 = \begin{pmatrix} 0 & 3 & 1 & 5 \\ & 0 & 1 & 8 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 3 & 1 & 5 \\ & 0 & 1 & 8 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 2 & 1 & 5 \\ & 0 & 1 & 7 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix} \quad M_4 = \begin{pmatrix} 0 & 2 & 1 & 5 \\ & 0 & 1 & 7 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix}$$

Note: M_k has same row/col k as M_{k-1}.

Floyd-Warshall Algorithm

Find all distances using the Floyd-Warshall algorithm.



Dijkstra vs Floyd-Warshall

To find distances for all pairs of vertices, we need to run Dijkstra's algorithm n times $\rightsquigarrow \mathcal{O}(n^3)$.

Floyd-Warshall is also $\mathcal{O}(n^3)$, but is quicker for large graphs.

One advantage to Floyd-Warshall is that it even works with negative edge weights.

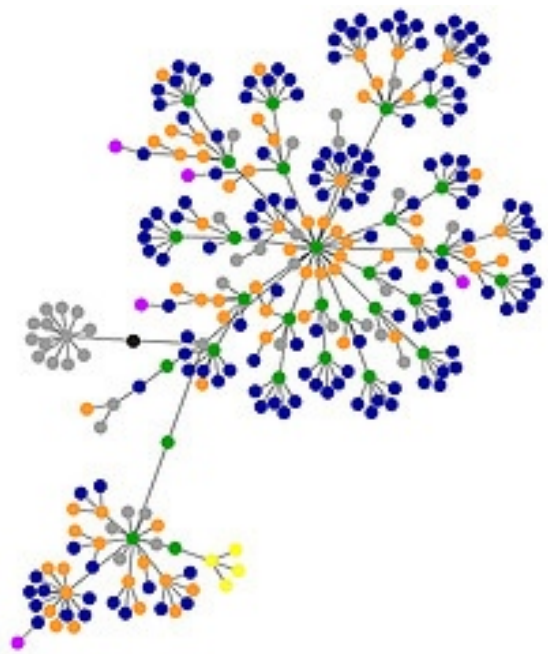
12 Trees

12.1 Trees and Their Properties

Trees

A circuit in a graph is a walk that starts and ends at the same point and traverses each edge at most once.

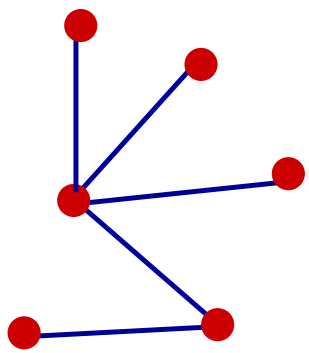
A tree is a connected graph with no circuits.



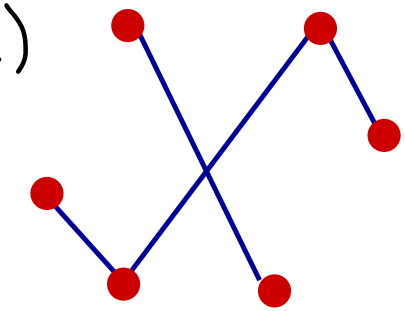
TREES

Which of the following graphs are trees?

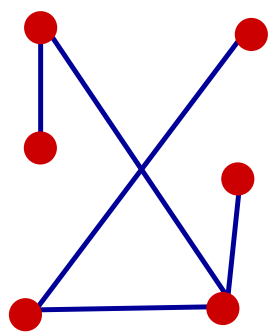
(i)



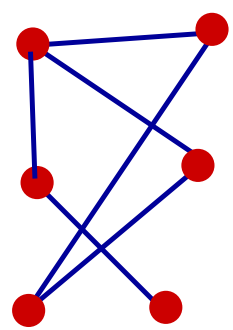
(ii)



(iii)

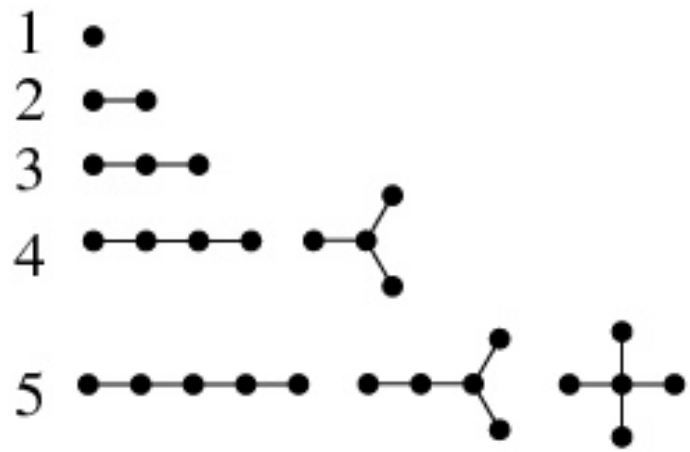


(iv)

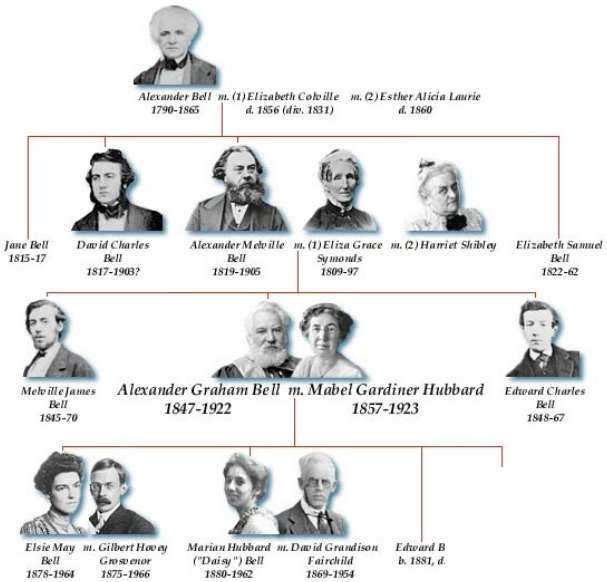


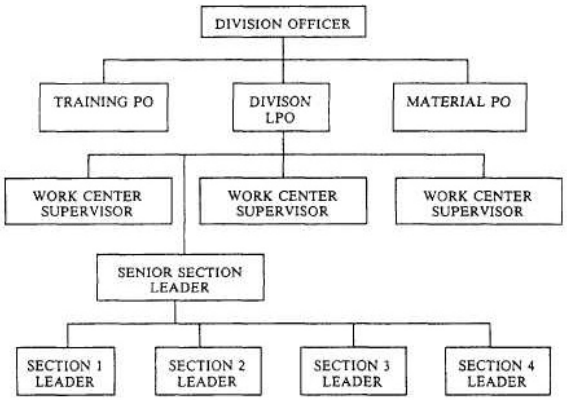
Trees

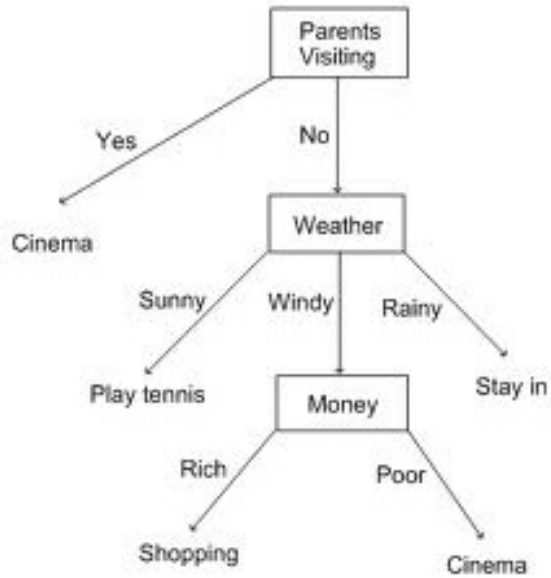
List all trees with 5 or fewer vertices up to isomorphism.

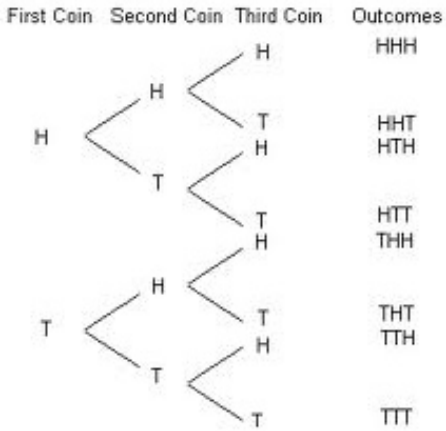


Applications of Trees









and many more...

Characterizing Trees

Theorem. Let G be a graph with n vertices. The following are equivalent:

(i) G is a tree (i.e. G is connected with no circuits)

(ii) G is connected and has no cycles.

(iii) G is connected and has $n-1$ edges.

(iv) Between any two vertices of G there is a unique walk that does not repeat any edges.

Also:

(v) G has $n-1$ edges and no cycles

(vi) G is connected, but removing any edge makes it disconnected.

(vii) G has no cycles, but adding any edge creates one.

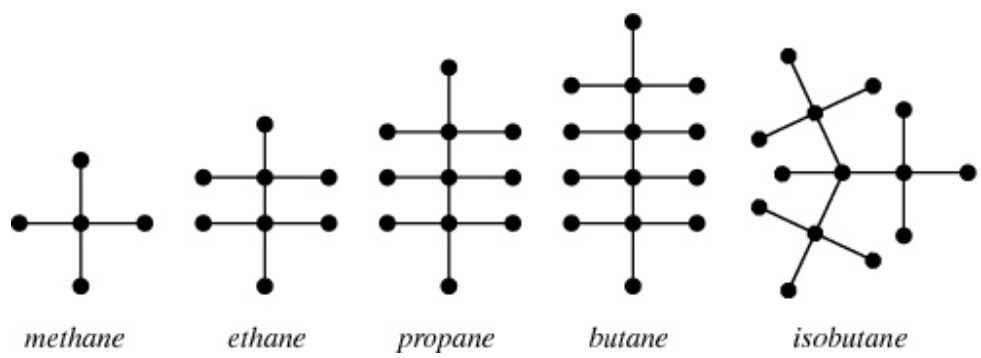
etc...

Application To Chemistry

A hydrocarbon has the form C_nH_{2n+2}. Carbon has degree 4 and Hydrogen has degree 1.

So we get a graph with $n+2n+2=3n+2$ vertices and $\frac{2n+2+4n}{2}=3n+1$ edges $\rightsquigarrow$ tree!

Problem. Find all hydrocarbons for $n=1,2,3,4$.



Characterizing Trees

Theorem. Let G be a graph with n vertices. The following are equivalent:

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(ii) G is connected and has no cycles.

(iii) G is connected and has $n-1$ edges.

(iv) Between any two vertices of G there is a unique walk that does not repeat any edges.

Sketch of Proof. (i)$\Leftrightarrow$(ii) Show, for any graph: cycle $\Leftrightarrow$ circuit

Idea: prune a circuit to get a cycle.

(i)$\Rightarrow$(iii) Show that no circuits $\Rightarrow$ degree 1 vertex.

Then use induction on n.

(i)$\Rightarrow$(iv) Given two different paths, construct a circuit.

(iv)$\Rightarrow$(ii) In a cycle, can find two paths b/w same vertices.

(iii)$\Rightarrow$(ii) Remove k edges to get a tree. Show $k=0$. $\blacksquare$

12.2 Spanning Trees

Spanning Trees

A spanning tree for a graph G is a subgraph that is a tree and that contains every vertex.

Note: Only connected graphs have spanning trees.

If we have a subgraph H of a weighted graph G, the weight of H is the sum of the weights of its edges.

A minimal spanning tree for a weighted graph is a spanning tree of least weight.

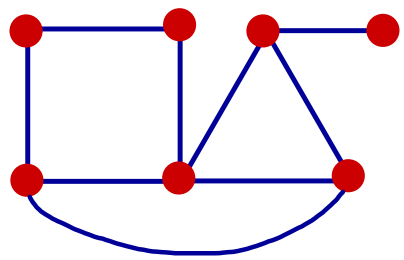
Application: Given a network of roads, which roads should you pave so that (a) all towns are connected and (b) we use the least amount of asphalt?

Spanning Trees

How to find a spanning tree?

One answer: Delete all edges until there are no cycles.

Example. How many spanning trees can you find?

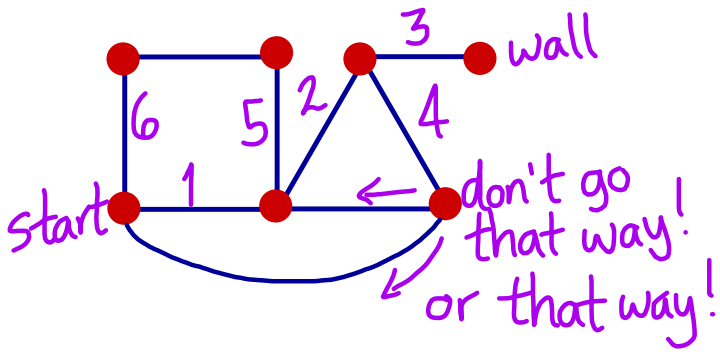


Question. How to find all spanning trees? How many are there?

Could hunt for cycles, delete edges. Inefficient!

Depth-First Search and Breadth-First Search

Depth-first: Start at some point in the graph.
Draw a long path, go as far as possible.
When you hit a wall (= degree 1 vertex),
or an edge that creates a cycle with your
path, back up one step and go in a new direction.



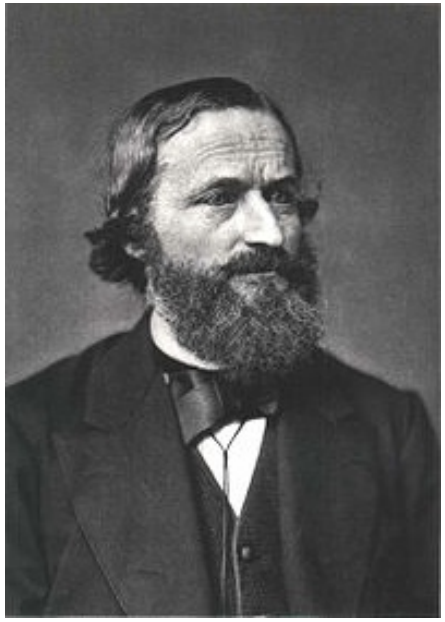
Breadth-first: Use as many edges from start point as
possible. Then from the endpoints of all
those edges use as many edges as possible, etc.

Kirchhoff's Theorem

Given a graph with vertices $v_1, \ldots, v_n$, make a matrix M with (i,i)-entry the degree of v_i and all other (i,j)-entries given by: -1 if $v_i v_j$ is an edge
0 otherwise

Theorem. Given a graph G, make the matrix M as above. Delete the ith row and the jth column to obtain a matrix M'. Then:

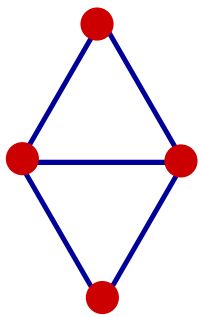
$$(-1)^{i+j} \det(M') = \#\text{ spanning trees for } G.$$



Gustav Kirchhoff

Kirchhoff's Theorem

Example.



$$\rightsquigarrow \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

$$+\begin{vmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= 3 \cdot 4 + 1 \cdot (-2) - 1 \cdot 2$$

$$= 8.$$

What are the 8 spanning trees? Find them all!

12.3 Minimal Spanning Tree Algorithms

Krushkal's Algorithm

Goal: Find a minimal spanning tree for a given graph.

Want something more efficient than enumerating all trees.

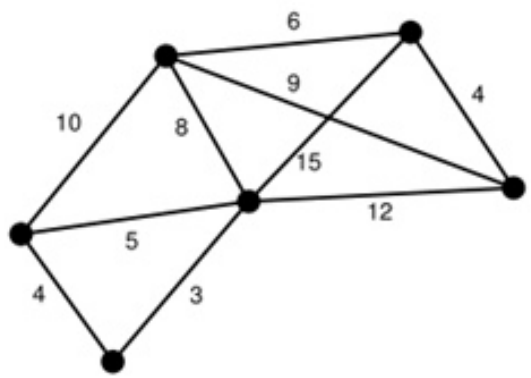
The Algorithm. Set $T = \emptyset$.
Consider all edges e so $T \cup \{e\}$ has no circuits.
Choose the edge e of smallest weight with this property.
Replace T with $T \cup \{e\}$.
Repeat until T is a spanning tree.

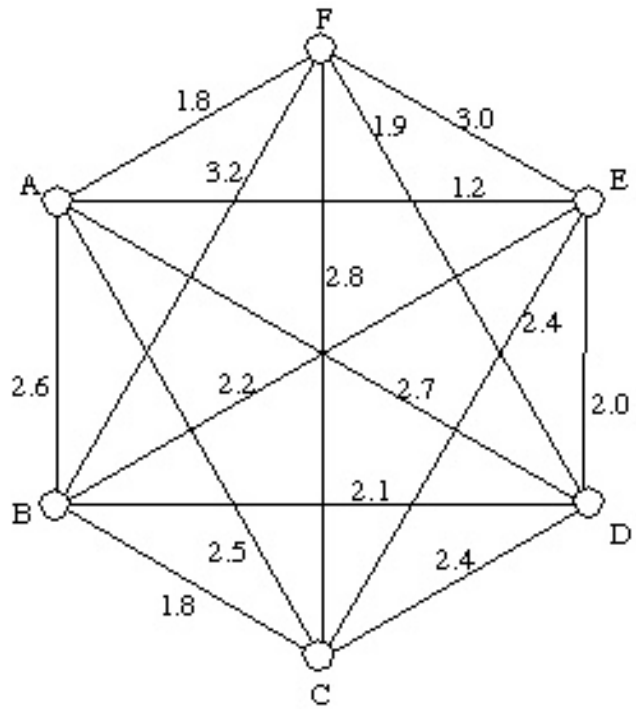
Note: The number of steps is one less than the # of vertices.

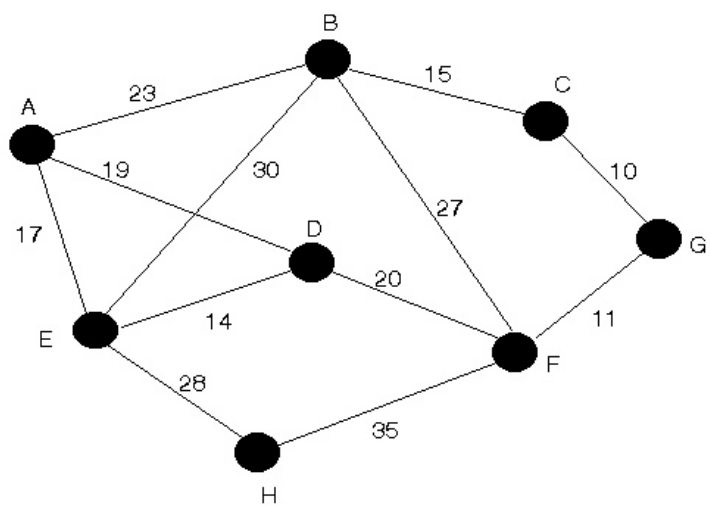
Krushkal's algorithm is an example of a "greedy algorithm"

Krushkal's Algorithm

Find minimal spanning trees for the following weighted graphs.







Krushkal's Algorithm

Why does the algorithm work?

Let $e_1, \ldots, e_{n-1}$ be the edges chosen by Krushkal's algorithm, in order.

Prove the following statement by induction:

$\{e_1, \ldots, e_k\}$ is contained in some minimal spanning tree.

Base case: $k=0$, i.e. $\emptyset$ contained in some minimal spanning tree. ✓

Suppose $\{e_1, \ldots, e_k\}$ contained in some minimal spanning tree T, but e_{k+1} is not in T. $\rightsquigarrow$ $T \cup e_{k+1}$ has a cycle.
There is an edge f contained in this cycle that is not equal to $e_1, \ldots, e_{k+1}$ (the e_i form a tree, so they form no cycles).
Now, f and e_{k+1} have same weight, otherwise weight of $T - f + e_{k+1}$ is less than weight of T. We see $T - f + e_{k+1}$ is the desired tree. ▨

Prim's Algorithm

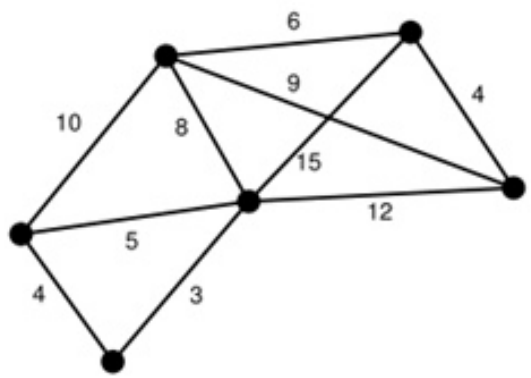
Idea: Grow a tree from a vertex.

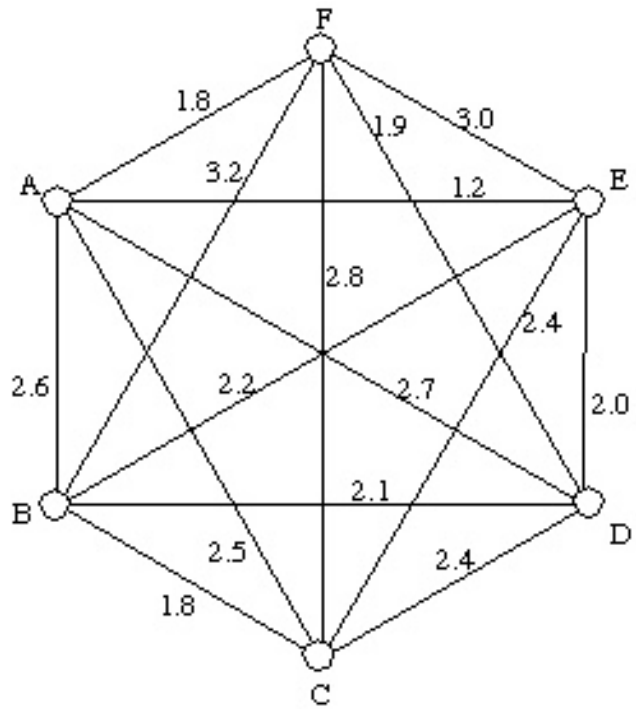
The algorithm. Set $T = v$ (any vertex)
Choose an edge e of minimal weight so $T \cup \{e\}$ is a tree
Replace T with $T \cup \{e\}$.
Repeat until T is a spanning tree.

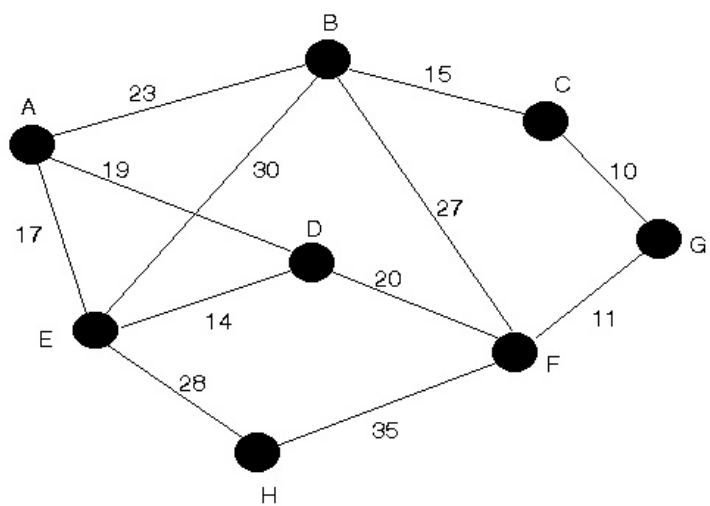
Note: We know $T \cup \{e\}$ is a tree if $T \cap e$ is a single vertex.

Prim's Algorithm

Find minimal spanning trees for the following weighted graphs.







Krushkal's Algorithm vs. Prim's Algorithm

What is the complexity? size = # edges
cost = # comparisons

Krushkal: $\mathcal{O}(n \log n + n^2)$

Prim: $\mathcal{O}(n^2)$

Check these! Idea: order the remaining edges. Then, need to check which can be added to the current tree by comparing the endpoints of each edge with the vertices of the current tree.

The advantage over Krushkal's algorithm is that there are fewer edges to check at each step. In fact, Prim is $\mathcal{O}(n^2)$.

Chapter 13
Planar Graphs and Colorings

13.1 Planar Graphs

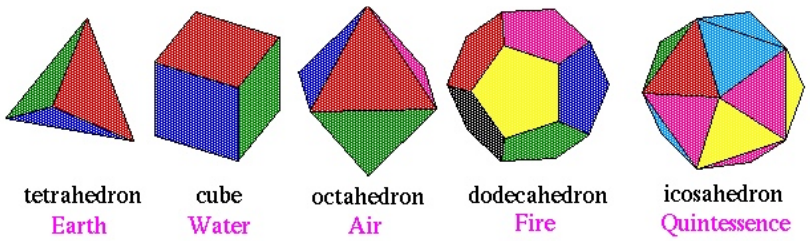
Planar Graphs

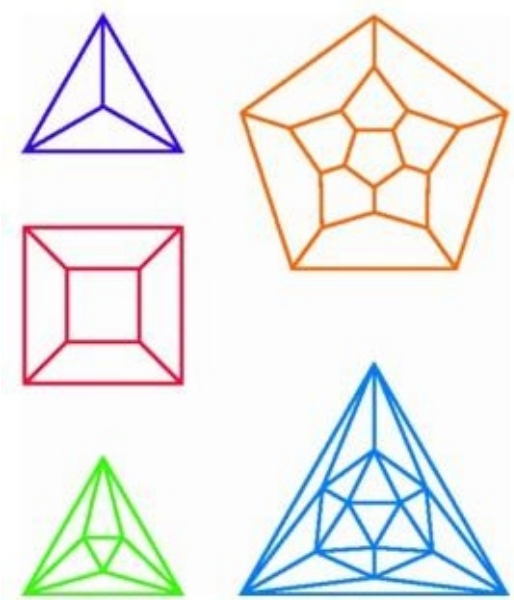
A graph is planar if it can be drawn in the plane so that no two edges cross.

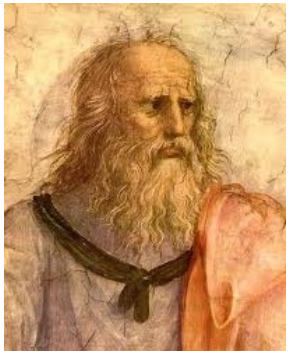
The Three House-Three Utility Problem asks whether or not $K_{3,3}$ is planar.

Platonic Solids

One collection of interesting planar graphs comes from the five Platonic solids:



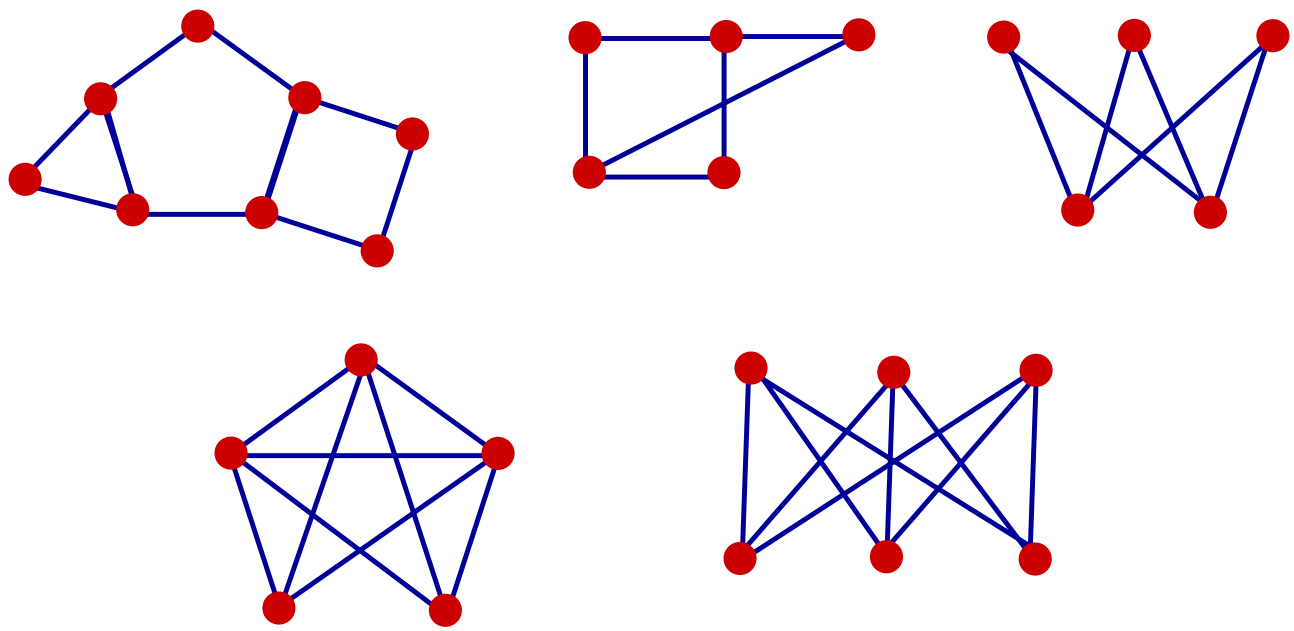




Plato

Planar Graphs

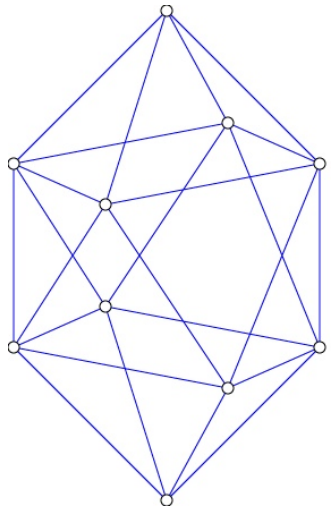
Which of the following graphs are planar?

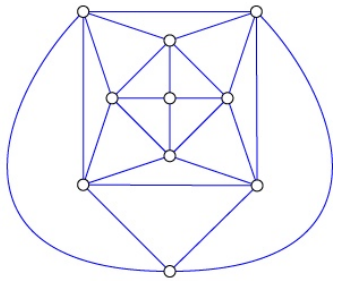


Note: First translate each graph from a picture of a graph to an abstract graph.

Planar Graphs

To show that a graph is planar, you just need to draw it in the plane with no crossings:



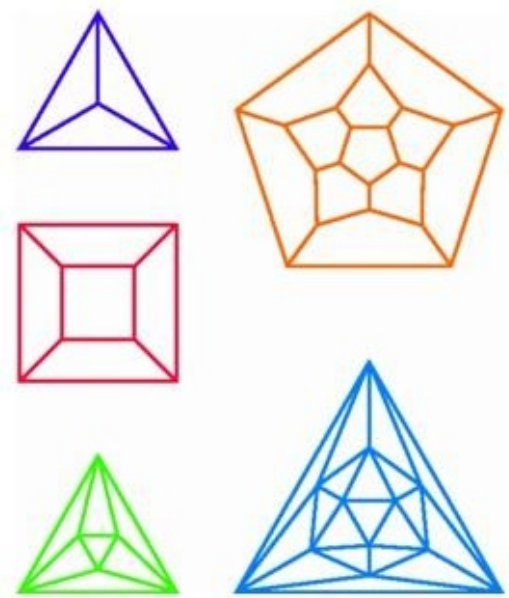


But how do we show a graph is <u>not</u> planar? For example, what about $K_{3,3}$? Is it possible to try all possible drawings? How many ways are there to draw $K_{2,2}$ or $K_{3,2}$ without crossings? Is there a better way?

Vertices, Edges, and Faces

A planar drawing of a planar graph divides the plane into distinct regions, or faces.

	vertices	edges	faces
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			



What is the pattern?

Euler's Theorem

Theorem. Any planar drawing of a graph with V vertices, E edges, and F faces satisfies

$$V - E + F = 2$$

In 1988, the Mathematical Intelligencer ran a survey. It was decided that the 5 most beautiful results in mathematics were:

(i) Euler's identity $e^{ix} = \cos x + i \sin x$
(ii) Euler's polyhedral formula $V - E + F = 2$
(iii) Euclid's proof of the infinitude of the primes
(iv) Euclid's proof that there are only 5 regular solids
(v) Euler's summation $\sum 1/n^2 = \pi^2/6$

Euler's Theorem

Theorem. Any planar drawing of a connected graph with V vertices, E edges, and F faces satisfies

$$V - E + F = 2$$

Proof. Induction on E.

Base case: $E = 0$ $\quad 1 - 0 + 1 = 2$ ✓

Assume the theorem is true for graphs with $E-1$ edges.

Case 1: G is a tree.

$\rightsquigarrow E = V - 1,\ F = 1 \rightsquigarrow V - E + F = V - (V-1) + 1 = 2.$

Case 2: G is not a tree

$\rightsquigarrow G$ has a cycle.

Let e be an edge of G in some cycle.

Then $G - e$ has V vertices, $E-1$ edges, $F-1$ faces.

By induction $V - (E-1) + (F-1) = 2$

$\rightsquigarrow V - E + F = 2$ $\blacksquare$

Platonic Solids

A Platonic solid is a 3-dimensional solid with polygonal faces, and satisfying:

(i) The faces are regular and congruent.
(ii) The same number of faces meet at each vertex.
(iii) The line connecting any two points on the solid is contained in the solid.

Theorem. There are exactly 5 Platonic solids.

Proof. Say we have a Platonic solid whose faces are n-gons, m at a vertex. $\rightsquigarrow$ Get a planar graph with

$$E = nF/2 \quad V = nF/m \qquad V - E + F = 2$$

$$\rightsquigarrow \quad 2E/m - E + 2E/n = 2$$

$$1/m + 1/n = 1/2 + 1/E$$

Notice $m, n \geq 3$. But by the above equation, m & n can't both be greater than 3... ▨

$K_{3,3}$ IS NOT PLANAR

THEOREM. $K_{3,3}$ is not planar.

PROOF. Suppose it is planar with F faces.

$\rightsquigarrow V - E + F = 6 - 9 + F = 2 \rightsquigarrow F = 5.$

Let N = sum of boundary edges of each region.

Note $N \leq 2E = 18$ since each edge is used at most twice.

But each face must have at least 4 sides, since $K_{3,3}$ has no triangles $\rightsquigarrow N \geq 5 \cdot 4 = 20$. Contradiction. ▨

K_5 is not Planar

Theorem. If a planar graph has V vertices and E edges, then $E \leq 3V-6$.

Proof. We may assume G is connected. Why?

If $V=3$ then $E \leq 3$ ✓

Now assume $V \geq 4$, $E \geq 3$.

Let N be as before. Again $N \leq 2E$

Also $N \geq 3F$ since each face has at least 3 sides.

$\rightsquigarrow 3F \leq 2E$

$$6 = 3V-3E+3F \leq 3V-3E+2E = 3V-E$$

$\rightsquigarrow E \leq 3V-6$ ∎

Corollary. K_5 is not planar.

Proof. $V=5$, $E = \binom{5}{2} = 10 > 3V-6 = 9$. ∎

Degrees

Theorem. Every planar graph has at least one vertex whose degree is less than 6.

Proof. Say all degrees are ≥ 6.

$2E = \text{sum of degrees} \geq 6V$

$\rightsquigarrow E \geq 3V > 3V - 6.$ $\blacksquare$

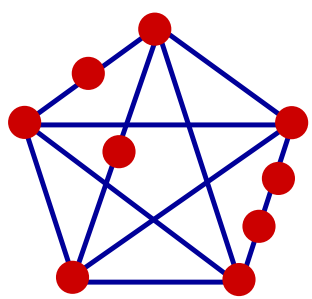
More Nonplanar Graphs

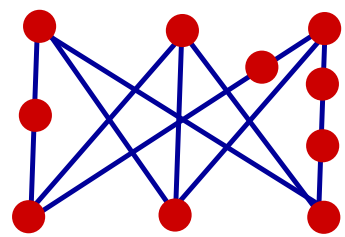
So far, we know K_5 and K_{33} are not planar.
It follows that K_n is not planar for $n \geq 5$,
$K_{m,n}$ is not planar for $m, n \geq 3$.

More generally:

Proposition. Any graph that contains K_5 or $K_{3,3}$ as a subgraph is not planar.

Note also any subdivision of K_5 or K_3 is nonplanar:





Proposition. Any graph that contains a subdivision of K_5 or $K_{3,3}$ as a subgraph is not planar.

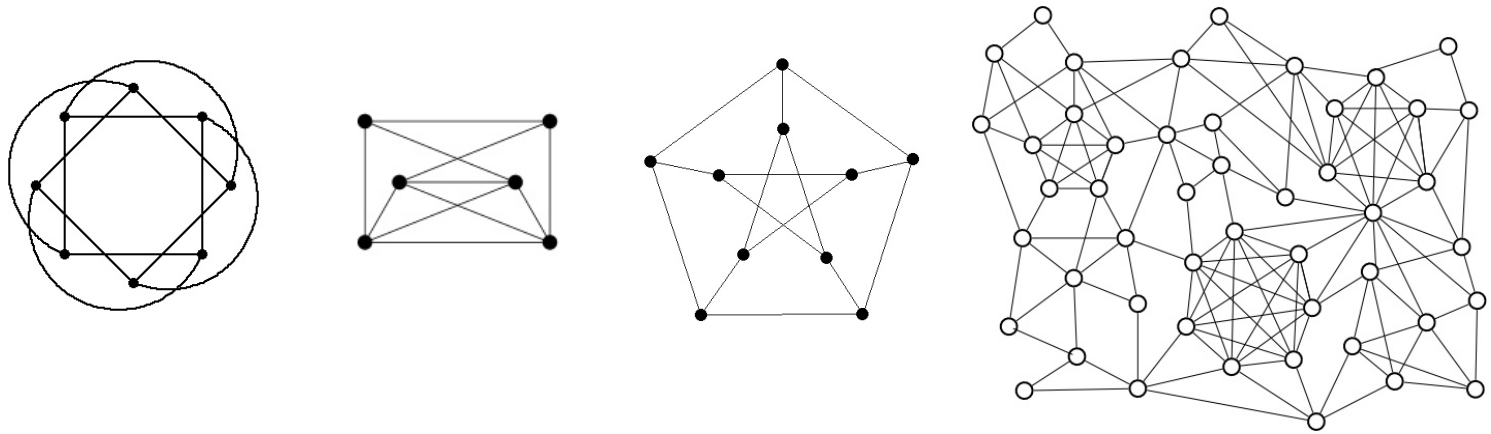
Kuratowski's Theorem

Amazingly, the converse is also true:

Theorem. A graph is planar if and only if it contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.

Proof. See web site.

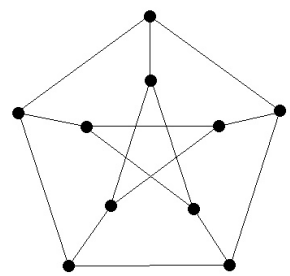
Which of the following graphs are planar?



Wagner's Theorem

A graph H is a minor of a graph G if H is obtained from G by taking a subgraph and collapsing some edges.

Theorem. A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.



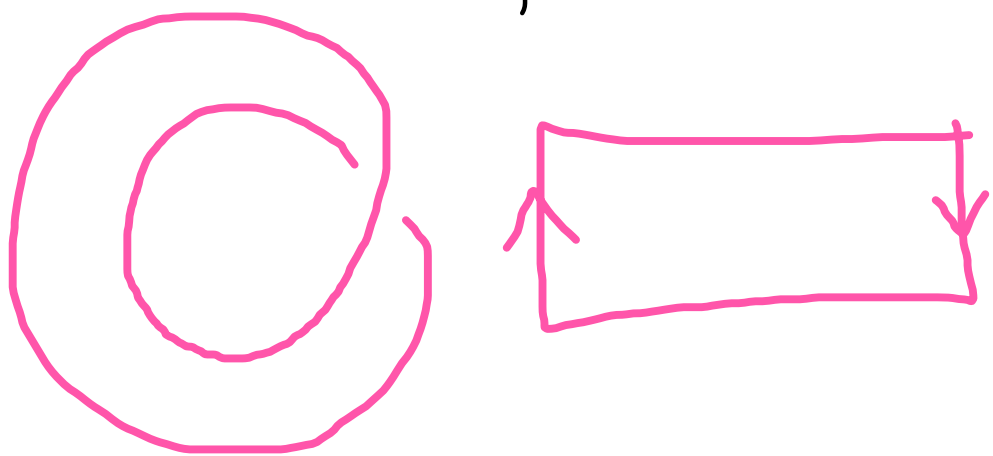
FÁRY'S THEOREM

THEOREM. Every planar graph can be drawn in the plane using only straight lines.

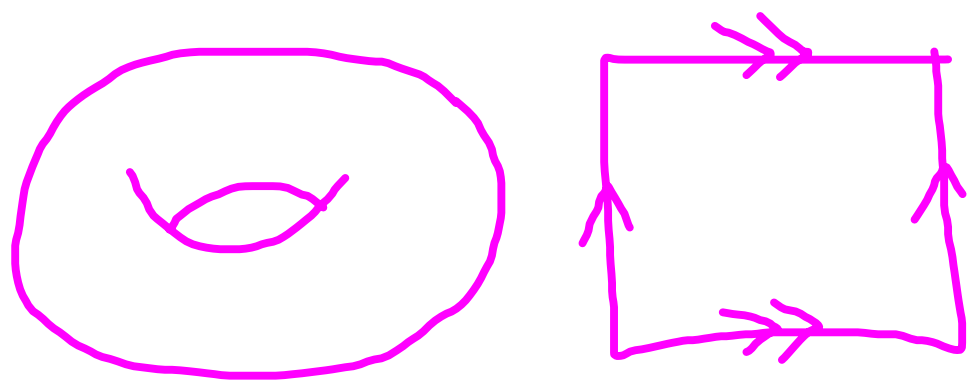
The proof uses the art gallery theorem...

Other Surfaces

What are the largest m, n so K_n and $K_{m,n}$ can be drawn without crossings on a Möbius strip



or a torus?



13.2 Coloring Graphs

The Four Color Problem

Show that, given any map in the plane, you can color it with four colors so that adjacent regions have different colors.

Notes. (i) Each region must be a connected "blob".
(ii) "Adjacent" means the regions meet in a segment (not just a corner).

Why are these caveats needed?


Francis Guthrie

Is there a map that really requires 4 colors?

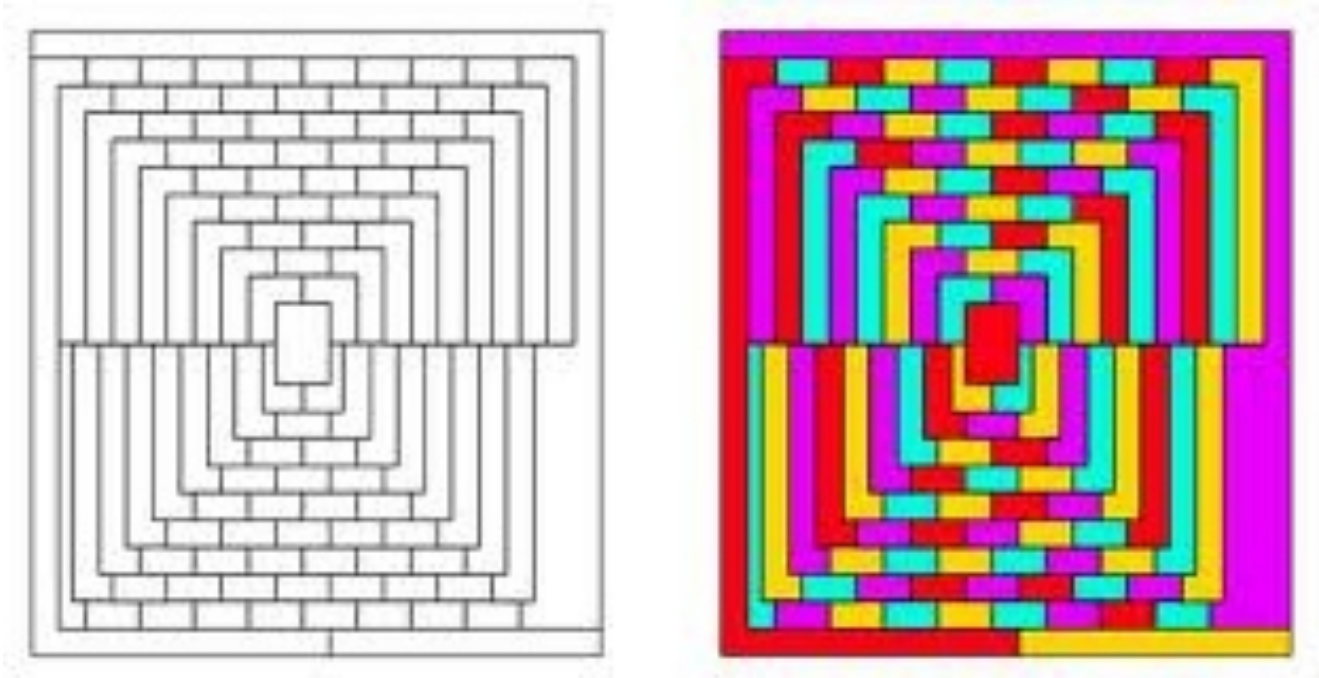
The Four Color Problem

How many colors are needed?

Hint: Look at Nevada.

The Four Color Problem

How many colors are needed?



For more challenges: nikoli.com

The Four Color Problem

First posed in 1852 by Guthrie. Many tried to solve it.
Alfred Kempe (1879) and Pether Guthrie Tait (1880) both gave solutions that stood for 11 years.

Lewis Carroll wrote about it:

> "A is to draw a fictitious map divided into counties.
>
> B is to color it (or rather mark the counties with names of colours) using as few colours as possible.
>
> Two adjacent counties must have different colours.
>
> A's object is to force B to use as many colours as possible. How many can he force B to use?"

The problem was solved in 1976 by Appel and Haken. It was the first major theorem proven in large part by computer.

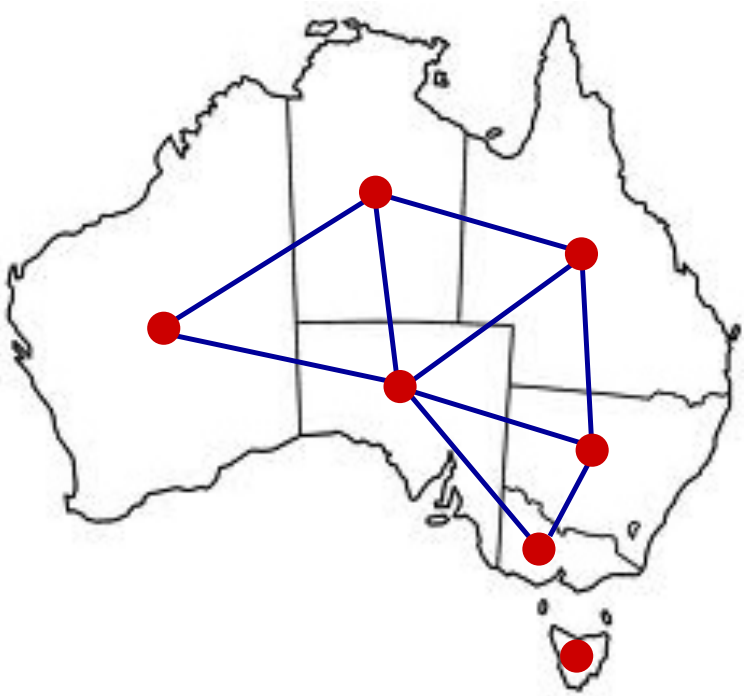
The proof has recently been simplified by Robin Thomas (GaTech) and his collaborators (still using computers).

Back to Graphs

Given a map, we get a graph $G(V,E)$ where

$V = \{\text{regions}\}$

$E = \{\text{pairs of adjacent regions}\}$



If the map is planar, then the graph is planar.

Coloring the map corresponds to coloring the vertices of the graph so that adjacent vertices have different colors.

Graph Coloring

A coloring of a graph is an assignment of colors to each of the vertices so that adjacent vertices have different colors.

The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed for a coloring of G.

Fact. $1 \leq \chi(G) \leq |V|$

Fact. If G is isomorphic to H, then $\chi(G) = \chi(G)$.

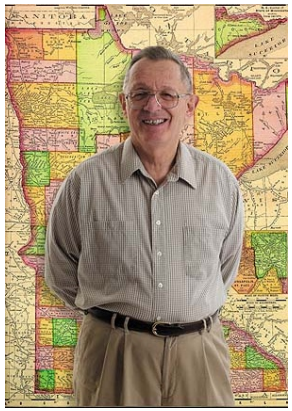
Fact. $\chi(K_n) = n$, $\chi(K_{m,n}) = 2$, and $\chi(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd.} \end{cases}$

Fact. If H is a subgraph of G then $\chi(H) \leq \chi(G)$

Fact. If G has a coloring with n colors, then $\chi(G) \leq n$.

The Four Color Theorem

Theorem. If G is planar, then $\chi(G) \leq 4$.



Kenneth Appel

Wolfgang Haken

Note: There is still no polynomial time algorithm for finding a coloring with 4 colors.

Applications

1. Sudoku. A vertex for each little square.
   An edge for two squares in same row, col, or 3×3 sqr.

2. Radio Frequencies. A vertex for each radio station.
   An edge between stations that are near eachother.

3. Scheduling. Example: Say there are 10 students taking

1 Physics, Math, IE	6 Physics, Geology
2 Physics, Econ, Geology	7 Business, Stat
3 Geology, Business	8 Math, Geology
4 Stat, Econ	9 Physics, Comp Sci, Stat
5 Math, Business	10 Physics, Econ, Comp Sci

What is the minimum number of final exam periods needed?

Six Colors Suffice

Proposition. If G is a planar graph then $\chi(G) \leq 6$.

Proof. Induction on the number of vertices.

Base case: one vertex ✓

Assume the proposition is true for planar graphs with $n-1$ vertices.

Let G be a planar graph with $n \geq 2$ vertices.

Recall that any planar graph has a vertex v of degree ≤ 5.

By induction, we can color $G - v$ with 6 colors.

Then color v differently from its neighbors using the sixth color. ▨

Degrees and Colors

Proposition. For any graph G:

$$\chi(G) \leq (\text{largest degree of a vertex of } G) + 1$$

Proof. Same as above.

Computing χ

To show that $\chi(G) = n$, we generally have to show two things:

① $\chi(G) \leq n$

Some possible reasons:

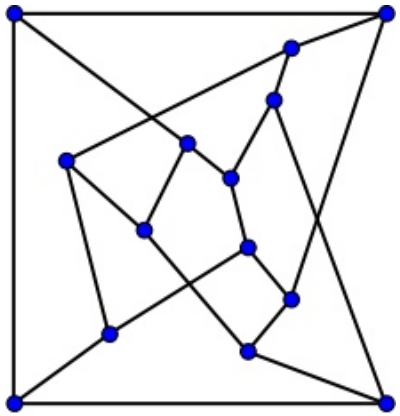
- G has n vertices
- G is bipartite
- G is planar
- Largest vertex degree is $n+1$
- We know an explicit coloring with n vertices.

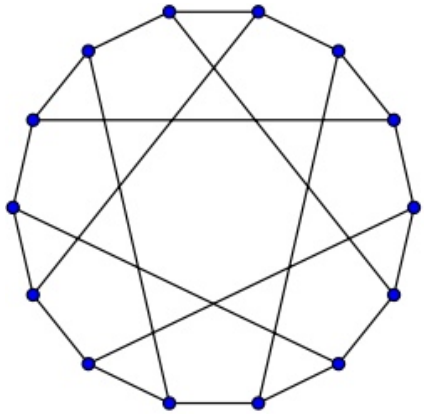
② $\chi(G) \geq n$

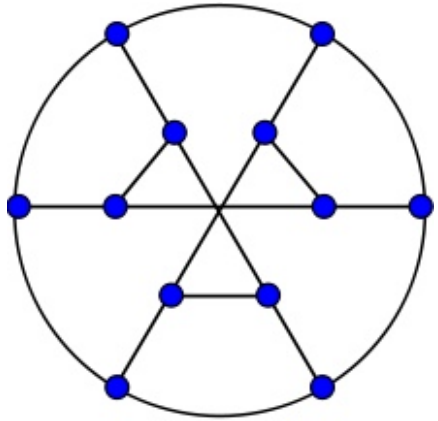
Some possible reasons:

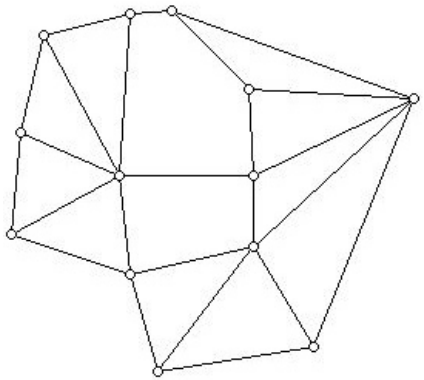
- G contains H and $\chi(H) = n$
- G contains H with $\chi(H) = n-1$ and a vertex adjacent to each vertex of H (cf. Nevada)

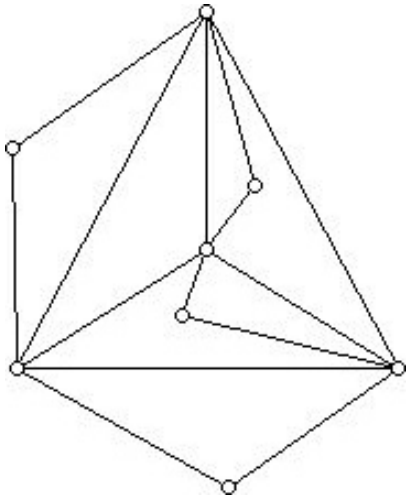
More Coloring Problems

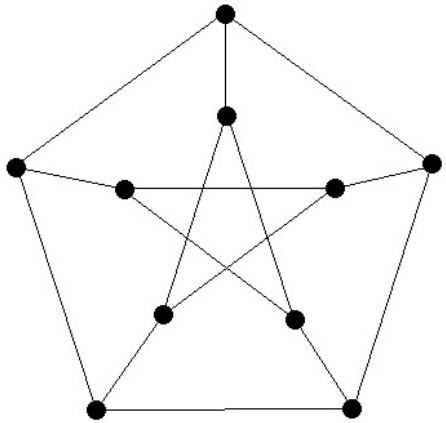










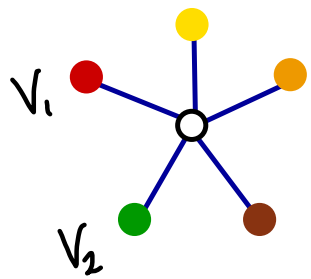


Five Colors Suffice


Percy Heawood

Theorem. If G is a planar graph, then $\chi(G) \leq 5$.

Proof. Induction on # vertices again.
Say G is a planar graph with n vertices.
As before, delete a vertex v of degree ≤ 5. Color $G-v$ with 5 colors. Can we reinsert v?



Case 1. There is no path from v_1 to v_2 using only red and green vertices.
In this case, starting at v_1, swap red and green.
Then color v red.

Case 2. There is such a path. Similar. ■

2 Solving Linear Systems

2.1 Echelon Form of a Matrix

Solving Linear Systems

Solve $\begin{aligned} 3x+3y&=9 \\ 3x+y&=7 \end{aligned} \xrightarrow{\text{subtract}} \begin{aligned} 3x+3y&=9 \\ 2y&=2 \end{aligned} \xrightarrow{\text{solve}} y=2 \xrightarrow{\text{back sub}} x=2$

Goal: eliminate variables

Can compactify this information using matrices:

$$\begin{pmatrix} 3 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

Goal: create zeros.

Simplifying the matrices $\leftrightarrow$ simplifying the equations

Row Echelon Form

An $m \times n$ matrix is in reduced row echelon form if:

1. Any zero rows are at the bottom.
2. The first nonzero entry of a row is 1 (called a "leading 1")
3. A leading 1 lies to the right of all leading 1's above it.
4. If a column has a leading 1, all other entries in that column are zero.

4 means reduced.

Example.

$$\begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{aligned} a + 4b \quad\quad &= 0 \\ c \quad &= 2 \\ d \quad &= 4 \end{aligned}$$

It is easy to solve the corresponding linear system.

Row Echelon Form

échelon = rung of a ladder.

Originally used to describe a formation of troops:

Row Echelon Form

Which matrices are in reduced row echelon form?

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If not, which criteria do they fail?

If a matrix is not in reduced row echelon form, can we make it so?

Row Operations

An elementary row operation on a matrix is any of:

Type I : interchange any two rows $\quad r_i \leftrightarrow r_j$

Type II : multiply a row by a number $\quad kr_i \to r_i$

Type III : add a multiple of one row to another $\quad kr_i + r_j \to r_j$

Examples:

$$\begin{pmatrix} 4 & 5 & 7 & 9 \\ 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{3r_2 \to r_2} \begin{pmatrix} 4 & 5 & 7 & 9 \\ 3 & 6 & 9 & 12 \\ 2 & 2 & 2 & 2 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 4 & 5 & 7 & 9 \\ 2 & 2 & 2 & 2 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

$$\xrightarrow{-2r_2 + r_1 \to r_1} \begin{pmatrix} 0 & 1 & 3 & 5 \\ 2 & 2 & 2 & 2 \\ 3 & 6 & 9 & 12 \end{pmatrix}$$

Row Equivalence

Two $m \times n$ matrices are row equivalent if one can be obtained from the other by a sequence of elementary row operations.

The matrices on the last page are all row equivalent.

Row equivalence is an equivalence relation:

(i) A is row equivalent to A.

(ii) If A is row equivalent to B then B is row equivalent to A.

(iii) If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

Reducing Matrices

Theorem. Every nonzero $m \times n$ matrix is row equivalent to a unique matrix in reduced row echelon form.

Recipe. Look at the first column with a nonzero entry.
Make that entry a 1 (Type II).
Move that 1 to the first row without a leading 1 (Type I)
Make all other entries in that column 0 (Type III).
Repeat: Find first column with nonzero entry below the last leading 1, ...

Examples. Find the reduced row echelon form:

$$\begin{pmatrix} 0 & 2 & 8 & -7 \\ 2 & -2 & 4 & 0 \\ -3 & 4 & -2 & 5 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 0 & 5 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{pmatrix}$$

2.2 Solving Linear Systems

Augmented Matrices

We solved $\begin{aligned} 3x+3y&=9 \\ 3x+y&=7 \end{aligned}$ via row operations on $\begin{pmatrix} 3 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$

We can go one step further and drop the x and y

$\rightsquigarrow$ augmented matrix $\left(\begin{array}{cc|c} 3 & 3 & 9 \\ 3 & 1 & 7 \end{array}\right)$

Theorem. If two augmented matrices differ by row operations, then the corresponding linear systems have the same solutions.

Homogeneous Systems

If the last column of an augmented matrix is the zero vector, the linear system is called homogeneous.

Arbitrary linear systems: $Ax=b$, e.g. $\begin{aligned} 3x+3y&=9 \\ 3x+y&=7 \end{aligned}$

Homogeneous linear systems: $Ax=0$, e.g. $\begin{aligned} 3x+3y&=0 \\ x+2y&=0 \end{aligned}$

In the homogeneous case, we can ignore the last column.

Homogeneous systems always have at least one solution.

Say A is an $n \times n$ matrix. The homogeneous system $Ax=0$ has a nonzero solution if and only if $\det(A)=0$ if and only if the last row of the row echelon form of A is all 0.

How Many Solutions?

The solution set to a system of linear equations can be:

(i) the empty set (no solutions)
(ii) a point (one solution)
(iii) a line (infinitely many solutions)
(iv) a plane (infinitely man solutions)
etc.

Say we are solving $Ax=b$.

We can easily see which case we are in by putting $(A|b)$ in reduced row echelon form.

How Many Solutions?

1. $$\begin{aligned} x + y \quad &= 0 \\ z &= 0 \\ 0 &= 1 \end{aligned} \longleftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

   no solutions

2. $$\begin{aligned} x \quad &= 0 \\ y \quad &= 0 \\ z &= 1 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

   one solution

3. $$\begin{aligned} x + \quad z &= 0 \\ y \quad &= 1 \\ 0 &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   ∞ many solutions:

   $y = 1$

   $x = -z$

   or: $(s, 1, -s)$

In general, variables that don't correspond to leading 1's are free.

How Many Solutions?

1. $x - 3y + z = 4$
   $2x - 8y + 8z = -2$
   $-6x + 3y - 15z = 9$

one solution:
$x = 3,\ y = -1,\ z = -2$

2. $2x - y + z = 1$
   $3x + 2y + 4z = 4$
   $-6x + 3y - 3z = 2$

no solutions (the first equation is almost a multiple of the third)

3. $x + y + z = 12$
   $3x - 2y + z = 11$
   $5x \quad + 3z = 35$

∞ many solutions:
$x = 7 - \frac{3}{5}z$
$y = 5 - \frac{2}{5}z$
or: $(7 - \frac{3}{5}s,\ 5 - \frac{2}{5}s,\ s)$

How Many Solutions?

4. $2x + 4y + 6z = 18$
$4x + 5y + 6z = 24$
$2x + 7y + 12z = 40$

5. $2x + 4y + 6z = 18$
$4x + 5y + 6z = 24$
$3x + y - 2z = 4$

6. $2x + 4y + 6z = 18$
$4x + 5y + 6z = 24$
$2x + 7y + 12z = 30$

7. $x + 2y + 3z = 6$
$2x - 3y + 2z = 14$
$3x + y - z = -2$

More Variables Than Equations

Theorem. If a system of linear equations has more variables than equations, then there are either no solutions or infinitely many.

In particular, if the system is homogeneous, there are infinitely many.
More specifically, the number of free parameters is the number of variables minus the number of equations.

Examples.

$$\begin{aligned} x_1 + x_2 + 2x_3 \quad - \tfrac{5}{2}x_5 &= \tfrac{2}{3} \\ x_4 + \tfrac{1}{2}x_5 &= \tfrac{1}{2} \end{aligned} \rightsquigarrow x_2, x_3, x_5 \text{ are free}$$

$$\begin{aligned} x + 3y - 5z + w &= 4 \\ 2x + 5y - 2z + 4w &= 6 \end{aligned} \rightsquigarrow \begin{aligned} x &= -2 - 19z - 7w \\ y &= 2 + 8z + 2w \end{aligned}$$

Linear Transformations

A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function
$T: \mathbb{R}^n \to \mathbb{R}^m$

with: (i) $T(kv) = kT(v)$ for $v \in \mathbb{R}^n$, $k \in \mathbb{R}$
(ii) $T(v+w) = T(v) + T(w)$ for $v, w \in \mathbb{R}^n$

$$\left\{ \begin{array}{c} \text{linear transformations} \\ \mathbb{R}^n \to \mathbb{R}^m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} m \times n \\ \text{matrices} \end{array} \right\}$$

- Given a linear transformation T we get a matrix whose column vectors are $T(e_1), \ldots, T(e_n)$

- Given a matrix M, we get a linear transformation
$T(v) = Mv$

example: $T(x,y) = (5x - 3y, x) \longleftrightarrow \begin{pmatrix} 5 & -3 \\ 1 & 0 \end{pmatrix}$

Linear Transformations

The *range* of a function $f: A \to B$ is $\{b \in B : b = f(a) \text{ for some } a\}$.

Problem. What is the range of the linear transformation associated to the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{pmatrix} ?$$

In other words, what are conditions on a, b, c so that

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

for some x, y, z?

Make the augmented matrix and try to solve.

Homogeneous versus Nonhomogeneous

Theorem. The set of solutions to $Ax = b$, $b \neq 0$, is

$$\{x_p + x_h\},$$

where x_p is any particular solution to $Ax = b$ and x_h ranges over all solutions to $Ax = 0$.

Compare with the case. In fact, prove the corresponding theorem about recurrence relations using this theorem.

Chapter 7

7.1 Eigenvalues and Eigenvectors

Eigenvalues And Eigenvectors

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

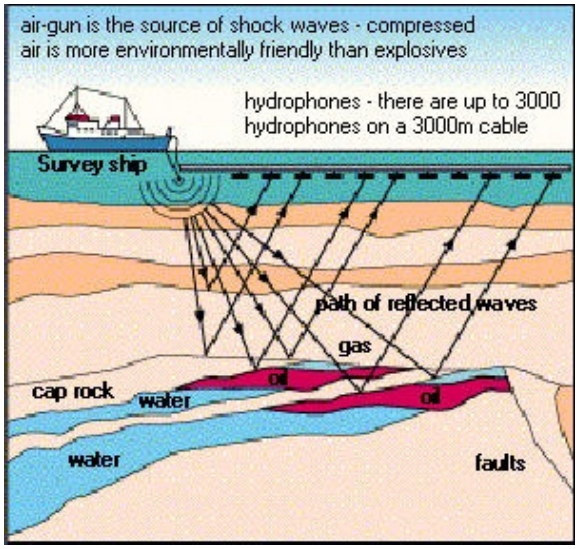
If there is a nonzero vector v and a real number λ such that

$$T(v) = \lambda v$$

then v is called an eigenvector for T and λ is called an eigenvalue for T.

Note: If v is an eigenvector then all nonzero multiples of v are eigenvectors $\rightsquigarrow$ line of eigenvectors

Applications



and many,

many more

Examples

1. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (-x, y)$ reflection over y-axis.

eigenvectors: $\{(0,y) : y \neq 0\}$ and $\{(x,0) : x \neq 0\}$
eigenvalues: $+1$ and -1.

2. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (2x, 2y)$ scale by 2.

eigenvectors: $\mathbb{R}^2 - \{0\}$
eigenvalues: 2

or: $T(x,y) = (3x, 2y)$

Examples

3. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ rotation by θ

eigenvectors: $\mathbb{R}^2 - \{0\}$ if $\theta = 0, \pi$, $\emptyset$ otherwise
eigenvalues: 1 or nothing.

4. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x,0)$ projection to x-axis.

eigenvectors: $\{(x,0) : x \neq 0\}$
eigenvalues: 1

Examples

5. $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (-x,-y,z)$ rotation by π about z-axis

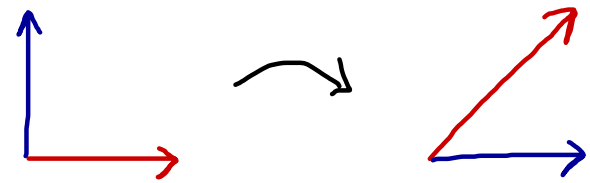
eigenvectors: $\{(x,y) \neq (0,0)\}$, $\{(0,0,z) : z \neq 0\}$
eigenvalues: -1 and 1

6. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (y,x)$ flip over $y=x$

eigenvectors: $\{(x,x) : x \neq 0\}$, $\{(x,x) : x \neq 0\}$
eigenvalues: 1 and -1

A More Complicated Example

7. $T(x,y) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x+y, x)$



What are the eigenvectors? Find them algebraically. Solve:

$$T(v) = \lambda v$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\left(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

There is a nonzero solution if and only if

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

or:

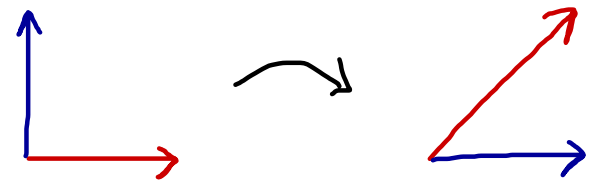
$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

A More Complicated Example

7. $T(x,y) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x+y, x)$



By the above calculation, the only possible eigenvalues are

$\lambda = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

Are there any eigenvectors with these eigenvalues? Solve:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightsquigarrow \begin{aligned} x + y &= \lambda x \\ x &= \lambda y \end{aligned} \rightsquigarrow \begin{pmatrix} \lambda y \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} (1+\sqrt{5})/2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} (1-\sqrt{5})/2 \\ 1 \end{pmatrix}$$

So we have eigenvectors of $\left(\frac{1+\sqrt{5}}{2}, 1\right)$ and $\left(\frac{1-\sqrt{5}}{2}, 1\right)$ with eigenvalues $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively.

Recipe for Finding Eigenvalues & Eigenvectors

Say A is an $n \times n$ matrix.

1. To find eigenvalues, solve

$$\det(A - \lambda I) = 0$$

Note: In general, eigenvalues are complex numbers.

2. For each eigenvalue λ solve

$$Av = \lambda v \quad \text{or} \quad (A - \lambda I)v = 0$$

The polynomial $\det(A - \lambda I)$ is called the characteristic polynomial of A. Its roots are exactly the eigenvalues of A.

Finding Eigenvalues & Eigenvectors

Find the eigenvalues and eigenvectors of the following matrices.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 10 & 6 \\ 6 & 4 \end{pmatrix} \quad \begin{pmatrix} 6 & 3 \\ -2 & -1 \end{pmatrix}$$

Solving the Characteristic Polynomial

Sometimes it is difficult to find the roots of the characteristic polynomial.

It sometimes works to guess roots. One strategy is to guess the divisors of the constant term (plus or minus).

Example. Find the eigenvalues and eigenvectors of:

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$$

Characteristic polynomial: $\lambda^3 - 6\lambda^2 + 11\lambda - 6$.
Guess: $\pm 1, \pm 2, \pm 3, \pm 6$ as roots...

Examples

$$\begin{pmatrix} 0 & 5 & 7 \\ -2 & 7 & 7 \\ -1 & 1 & 4 \end{pmatrix}$$ eigenvalues: $2, 4, 5$

$$\begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$ eigenvalues: $2, 3, 5$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$ eigenvalues: $2, -1, -1$

Google Page Rank

Say the internet has pages $P_1, \ldots, P_N$.

Denote the importance of the ith page P_i by $I(P_i)$.

To determine importance, each page gets 1 vote, split equally amongst outgoing links, BUT votes from important pages get more weight. (chicken and egg??):

$$I(P_i) = \sum_j I(P_j)/l_j \qquad (*)$$

l_j ← #links from P_j

How to compute I? Make a matrix:

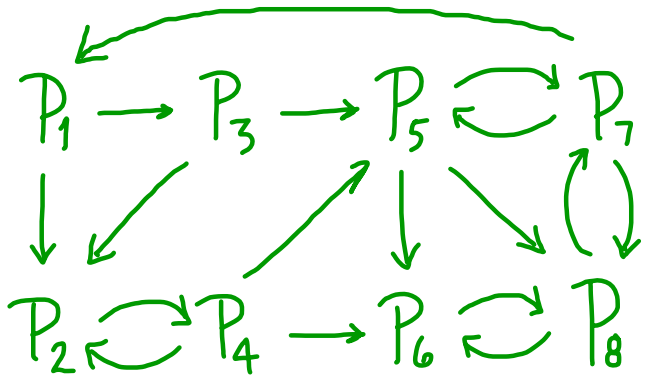
$$H_{ij} = \begin{cases} 1/l_j & P_j \text{ links to } P_i \\ 0 & \text{otherwise.} \end{cases}$$

Condition $(*)$ is equivalent to: $H(I) = I$.

⇝ the importance vector I is an eigenvector for H with eigenvalue 1.

Google Page Rank

Example.



$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/3 & 1 & 1/3 & 0 \end{pmatrix} \rightsquigarrow I = \begin{pmatrix} .06 \\ .07 \\ .03 \\ .06 \\ .09 \\ .20 \\ .18 \\ .31 \end{pmatrix}$$

$\rightsquigarrow$ So page 8 is most important.

Google Page Rank

In real life $N = 25$ billion!

How to find I?

Idea: Iterate. Take any vector v, say $v = e_1$.
The sequence $H^k(v)$ approaches (the line through) I.

In above example, $H^{60}(e_1) \sim I$.

Principle: A linear transformation pulls most vectors towards the (leading) eigenvector. See the next lecture!

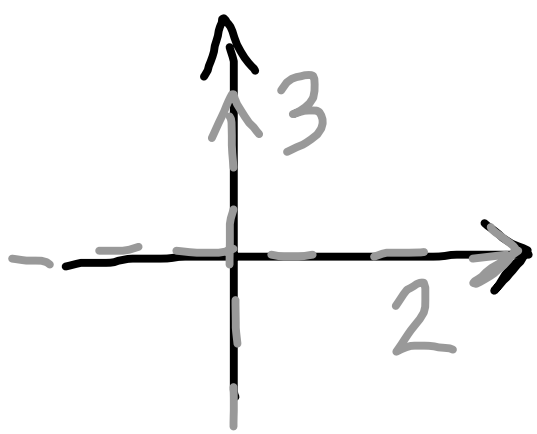
7.2 DIAGONALIZATION

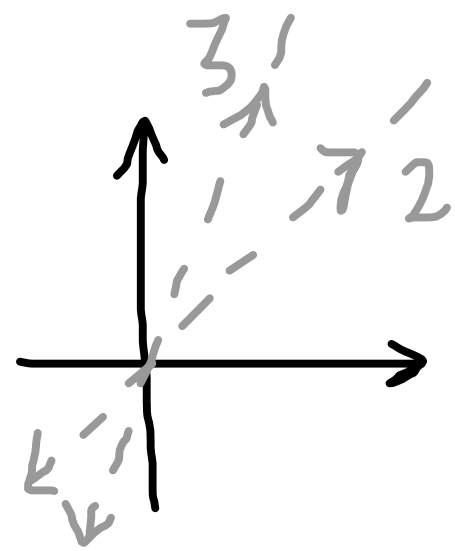
Diagonalizing Matrices

What does $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ <u>do</u> to $\mathbb{R}^2$?

We find eigenvectors: $(2,1)$ and $(1,1)$
eigenvalues: 2 $\quad$ 3

It is similar to $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$



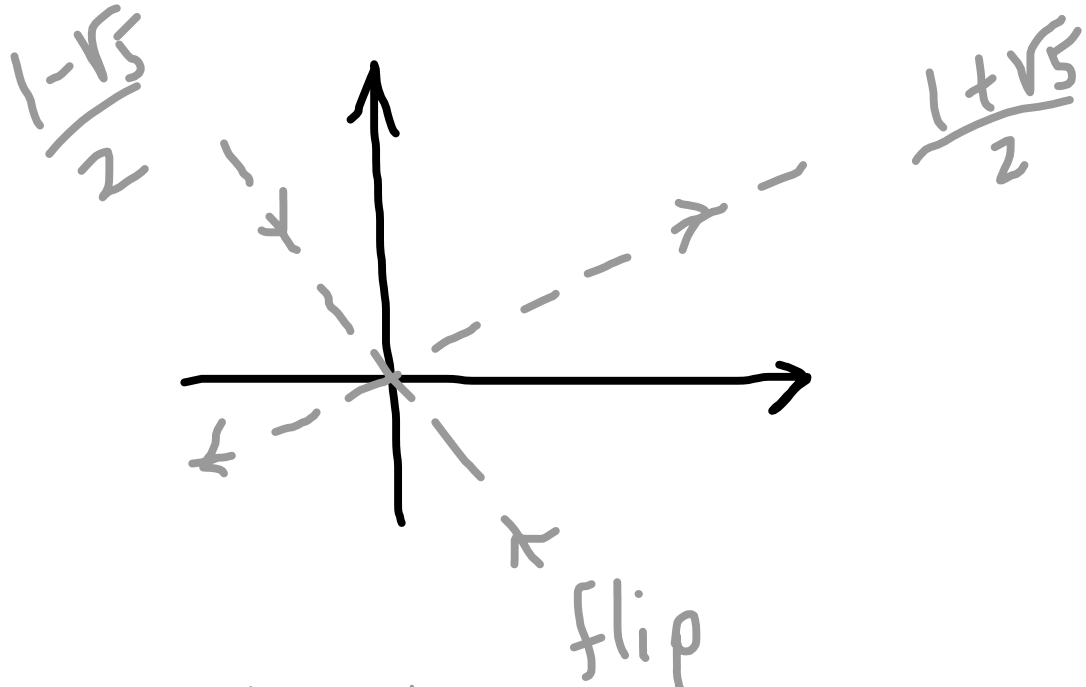


Similar means: doing the same thing, but with respect to different bases.

Diagonalizing Matrices

What about $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$?

similar to $\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$

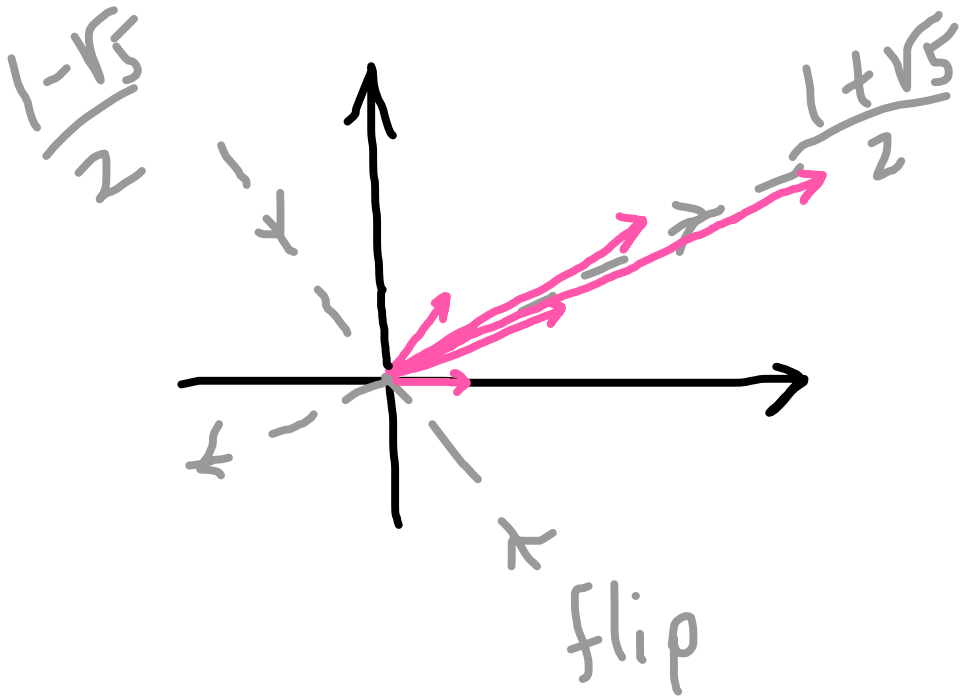


Similar means: doing the same thing, but with respect to different bases.

Diagonalizing Matrices

What about powers of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$?

similar to $\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^k \end{pmatrix}$



$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

etc.

We conclude:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

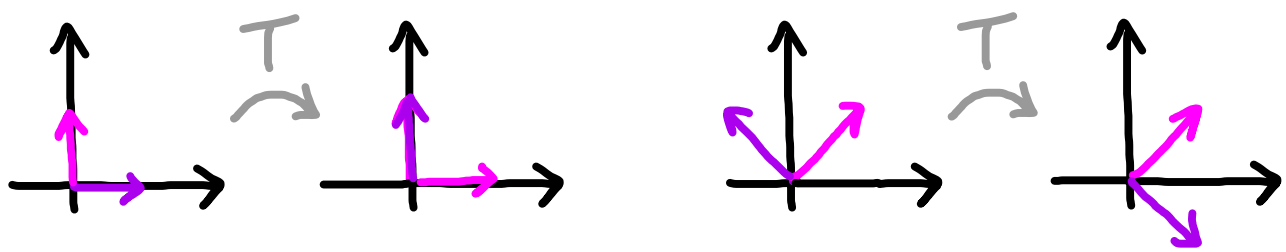
Similar Matrices

Two matrices A and B are similar if there is a matrix C so that

$$A = CBC^{-1}$$

This means that A and B are essentially the same, just written with respect to different bases.

Example. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be reflection over the line $y=x$. We write T with respect to two different bases:



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is the change of basis

Similar Matrices

Show that the following matrices are similar:

1. $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$

Hint: Write the preferred basis for one in terms of the preferred basis for the other, as in the previous example.

Use $C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$.

Diagonalizable Matrices

A matrix is diagonalizable if it is similar to a diagonal matrix.

If a matrix A is diagonalizable, it is easy to compute powers of A:

$$A = CDC^{-1}$$

$$\rightarrow \quad A^k = (CDC^{-1})^k$$

$$= CD^kC^{-1}$$

Computing D^k is a snap:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{pmatrix}$$

So finding A^{1000} only requires <u>two</u> matrix multiplications.

Diagonalizable Matrices

1. Compute $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^5$.

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^5 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^5\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32 & 0 \\ 0 & 243 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -179 & 422 \\ -211 & 454 \end{pmatrix}$$

2. We saw $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$. Use this to find an explicit formula for F_n. How does this relate to our old method?

Eigenvalues and Similarity

Theorem. Similar matrices have the same eigenvalues.

Proof. Say $B = CAC^{-1}$.

$$\det(B-\lambda I) = \det(CAC^{-1}-\lambda I) = \det(CAC^{-1}-\lambda CIC^{-1})$$
$$= \det(C(A-\lambda I)C^{-1}) = \det(C)\det(A-\lambda I)\det(C^{-1})$$
$$= \det(A-\lambda I).$$

Theorem. If a matrix A is similar to a diagonal matrix D, the eigenvalues of A are the same as the diagonal entries of D.

DIAGONALIZABLE?

How do we know if a matrix A is diagonalizable?

The algebraic multiplicity of an eigenvalue λ for A is the number of times λ appears as a root of the characteristic polynomial $\det(A-\lambda I)$.

Example. The algebraic multiplicity of 5 in $(\lambda-5)^2(\lambda-1)$ is 2.

The geometric multiplicity of an eigenvalue λ for A is the number of free parameters in the solution of $(A-\lambda I)v=0$.

This is the dimension of the eigenspace for λ.

THEOREM. A square matrix is diagonalizable if and only if each eigenvalue's algebraic and geometric multiplicities are equal.

Diagonalizable?

Theorem. A square matrix is diagonalizable if and only if each eigenvalue's algebraic and geometric multiplicities are equal.

Two restatements:

Theorem. An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.

Theorem. A matrix is diagonalizable if and only if each eigenvalue of multiplicity k has k linearly independent eigenvectors.

DIAGONALIZABLE?

THEOREM. A square matrix is diagonalizable if and only if each eigenvalue's algebraic and geometric multiplicities are equal.

We have:

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}$$

Therefore, if the algebraic multiplicity of λ is 1, there is nothing to check.

If all algebraic multiplicities are 1, there is <u>really</u> nothing to check:

COROLLARY. If an $n \times n$ matrix has n distinct eigenvalues, it is diagonalizable.

Diagonalizable?

1. Is $\begin{pmatrix} 2 & -3/2 \\ 0 & 1/2 \end{pmatrix}$ diagonalizable?

   Yes. Eigenvectors are $(1,0)$ and $(1,1)$.

2. Is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ diagonalizable?

   No. All eigenvectors on x-axis.

3. Is $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ diagonalizable?

   Yes. Two distinct eigenvalues.

Diagonalization Recipe

Say A is diagonalizable, so $A = CDC^{-1}$. How to find C and D?

- Put the eigenvalues of A in some order: $\lambda_1, \ldots, \lambda_n$.
- Choose n linearly independent eigenvectors $v_1, \ldots, v_n$, where the eigenvalue for v_i is λ_i.

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \qquad C = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

Then need to find C^{-1}.

Why this works: $AC = CD$

j^{th} col of $AC = A \cdot j^{th}$ col of C

j^{th} col of $CD = \lambda_j \cdot j^{th}$ col of C

Diagonalization Recipe

Diagonalize the following matrices:

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$$

Recall: To find C^{-1}, write

$$(C \mid I)$$

Row reduce:

$$(I \mid C^{-1})$$

Diagonalization

Are the following matrices diagonalizable? If so, diagonalize.

$$\begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

No.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

No.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Yes.

Linear Programming

Linear Programs

Example. Maximize $3x+2y = z$ ← objective function

subject to $\left.\begin{array}{l} 2x+y \leq 20 \\ x,y \geq 0 \end{array}\right\}$ ← constraints

e.g. x = widget, y = gadget
3 = profit on widget
2 = profit on gadget

2 hrs to make widget
1 hr to make gadget
20 hours available time

A maximization (or mimimization) problem where the objective function and constraints are all linear is called a linear programming problem.

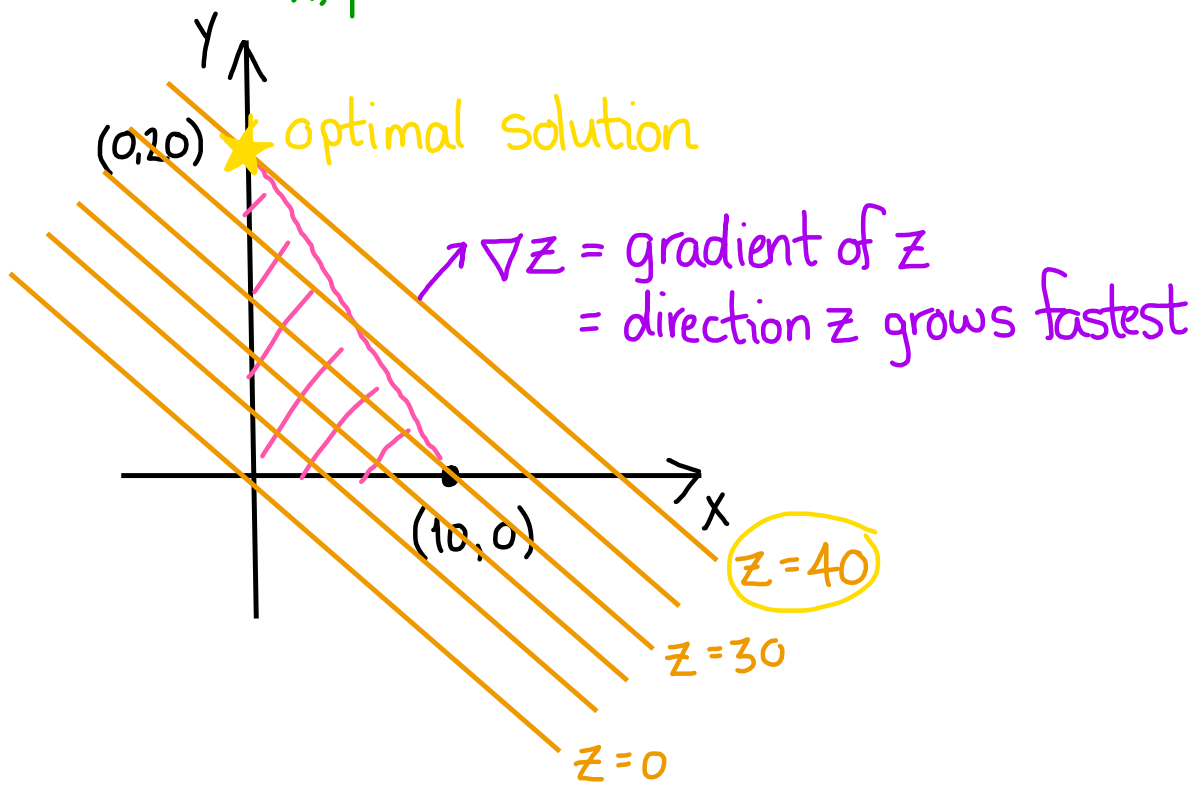
How to solve?

$3x+2y$ is linear
$(x+y)^3$, $3\ln(x)$, $\sqrt{x+y}$ are not linear

Linear Programs

Example. Maximize $3x+2y = z$ ← objective function

subject to $\left.\begin{array}{l} 2x+y \le 20 \\ x, y \ge 0 \end{array}\right\}$ ← constraints



∇z = gradient of z
= direction z grows fastest

Pink triangle = feasible region

Linear Programs

Example. Maximize $z = x + y$

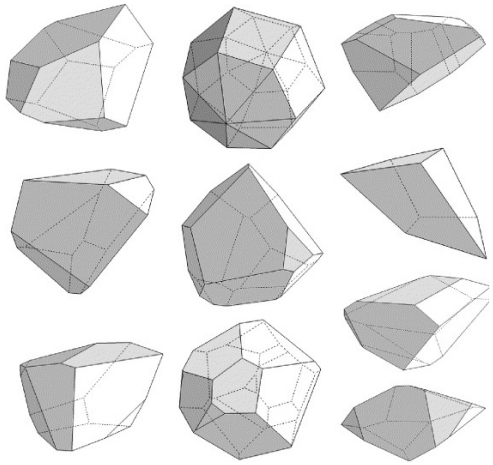
subject to
$$2x + 3y \leq 6$$
$$4x + 2y \leq 8$$
$$x, y \geq 0$$

Answer: $z = 6.5$

Notice: The optimum always occurs at a corner.

This always works! To find the optimum, we move the objective function hyperplane in the direction of its perpendicular (gradient) and observe the last point(s) of the feasible region it passes through. This will always be at a corner.

The Feasible Region



The feasible region for a linear program is the intersection of finitely many half-spaces. Thus, it is a convex (possibly infinite) polyhedron.

We deduce:

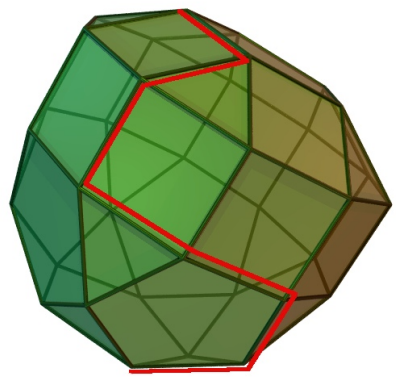
Fact. If a finite optimum exists for a linear program, then there is an optimal extreme point (= corner).

In other words, to find optima, it is enough to look at corners. But simply checking all corners takes way too long!

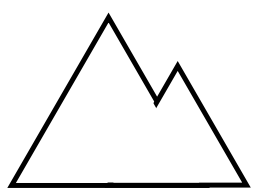
Example. Maximize $z = 3x$ s.t. $0 \le x, y \le 1$.

The Simplex Method

The basic idea: Start at some corner of the feasible region. See if any adjacent corners are higher (in z-value). If so, move to that corner. If not, stop.



In other words, if you always move up, you eventually get to the top. This does not work for nonconvex shapes:



Now: How to formalize this?

The Simplex Method

The simplex method was devised in 1947 by George Dantzig, of the RAND corporation.

George Dantzig

It was deemed one of the top ten algorithms of the 20th century in the Jan/Feb 2000 issue of Computing in Science and Engineering.

Standard Form

Given a linear program, we put it in standard form by adding slack (or surplus) variables so that all inequalities become equalities.

Example. The standard form of

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

is:

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 + x_3 = 4 \\ & x_1 + 2x_2 + x_4 = 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Standard Form

The standard form: maximize $z = x_1 + x_2$

subject to
$$\begin{aligned} 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

gives a system of linear equations:
$$\begin{aligned} z - x_1 - x_2 \quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

subject to the condition $x_i \geq 0$.

So we are looking at the intersection of an $(n-k)$-plane with the positive orthant of $\mathbb{R}^n$, assuming there are $n-1$ of the x_i and $k-1$ original constraints (not counting positivity).

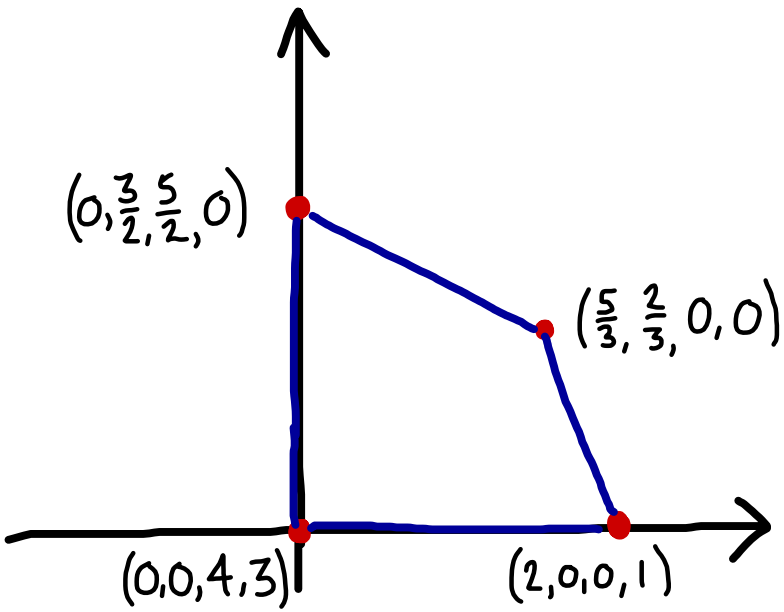
We usually draw the projection that kills the slack variables.

Standard Form

The picture of the feasible region for

$$
\begin{aligned}
z - x_1 - x_2 \qquad &= 0 \\
2x_1 + x_2 + x_3 \quad &= 4 \\
x_1 + 2x_2 \quad + x_4 &= 3 \\
x_i \geq 0.
\end{aligned}
$$

is:



The Simplex Method

After putting in standard form, we want to maximize z, where:

$$\begin{aligned} z - x_1 - x_2 \qquad\quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

and $x_i \geq 0$.

In these notes, we assume no surplus variables (i.e. all original constraints are $\leq$) and all right-hand sides are positive.

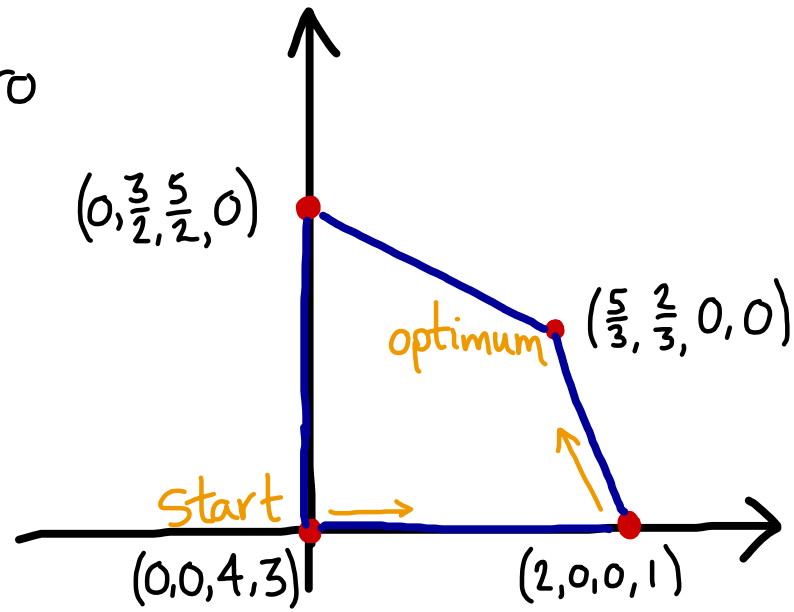
A basic variable is one that appears in only one equation.

Rule 0. Setting all nonbasic variables equal to 0 gives a corner of the feasible region, called a basic solution.

In the above example, the basic solution is $z = 0$ at $(0,0,4,3)$.

The Simplex Method

As we can see, by changing one nonzero coordinate to be zero, we move along an edge of the feasible region. So to make progress from our starting point to the optimal solution, we change the basic variables, one at a time.



We just need to make sure ① We are always increasing z
and ② We stay in the feasible region

Rules ① and ② below address these two points.

The Simplex Method

Say we have a linear program in standard form:

$$\begin{aligned} z - x_1 - x_2 \qquad\quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

Rule 1. The current basic solution is optimal if and only if all variables in the top row have nonnegative coefficients.

If the current basic solution is not optimal, we choose a variable with negative coefficient in the top row and make it basic using row operations.

But, we need to do this carefully!

In Rule 1, we usually choose a variable with <u>most</u> negative coeff.

The Simplex Method

Say we have a linear program in standard form:

$$\begin{aligned} z - x_1 - x_2 \qquad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

Say, by Rule 1, we decide to make x_1 basic. This means we want to use row operations to remove x_1 from all equations but one. Which to choose?

Rule 2. When making x_i basic, we leave x_i in the row where

$$\frac{\text{RHS}}{\text{coeff}(x_1)}$$

is the smallest positive number among all rows.

positive $\rightsquigarrow$ z will increase smallest $\rightsquigarrow$ stay in feasible region

The Simplex Method

Rules 1 and 2 comprise the simplex method. Let's apply it to our example.

$$\begin{aligned} z - x_1 - x_2 \qquad\quad &= 0 \\ 2x_1 + x_2 + x_3 \quad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

By Rule 1, we are not at the optimum, and we choose to make x_1 basic.

By Rule 2, we use the $2x_1$ as our pivot, and get:

$$\begin{aligned} z \quad - \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 \qquad &= 2 \\ x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 \qquad &= 2 \\ \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 &= 1 \end{aligned}$$

What happens if you use the bottom row as a pivot?

$\longrightarrow$ new basic solution $(2,0,0,1)$, $z=2$.

The Simplex Method

We now have:

$$\begin{aligned} Z \quad -\tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 \qquad &= 2 \\ x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 \qquad &= 2 \\ \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 &= 1 \end{aligned}$$

By Rules 1 and 2, we make x_2 basic by pivoting from the bottom row. We get:

$$\begin{aligned} Z \qquad + \tfrac{1}{3}x_3 + \tfrac{1}{3}x_4 &= 7/3 \\ x_1 \quad + \tfrac{2}{3}x_3 - \tfrac{1}{3}x_4 &= 5/3 \\ x_2 - \tfrac{1}{3}x_3 + \tfrac{2}{3}x_4 &= 2/3 \end{aligned}$$

$\rightsquigarrow$ basic solution $(5/3, 2/3, 0, 0)$ $Z = 7/3$.

This is optimal by Rule 1!

TABLEAUX

We can succinctly record (and perform) the above calculation as follows:

z	x_1	x_2	x_3	x_4	RHS	Basic Soln
1	-1	-1	0	0	0	
0	2	1	1	0	4	$(0,0,4,3)$ $z=0$
0	1	2	0	1	3	
1	0	$-1/2$	$1/2$	0	2	
0	1	$1/2$	$1/2$	0	2	$(2,0,0,1)$ $z=2$
0	0	$3/2$	$-1/2$	1	1	
1	0	0	$1/3$	$1/3$	$7/3$	
0	1	0	$2/3$	$-1/3$	$5/3$	$(5/3, 2/3, 0, 0)$ $z=7/3$
0	0	1	$-1/3$	$2/3$	$2/3$	

Note the basic solution is easily read from the RHS.

The Simplex Method

Problem. Maximize $4x_1 + x_2 - x_3 = z$

subject to
$$\begin{aligned} x_1 + \quad 3x_3 &\le 6 \\ 3x_1 + x_2 + 3x_3 &\le 9 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

First we write the standard form:

$$\begin{aligned} z - 4x_1 - x_2 + x_3 \qquad\qquad &= 0 \\ x_1 \qquad + 3x_3 + x_4 \qquad &= 0 \\ 3x_1 + x_2 + 3x_3 \qquad + x_5 &= 9 \end{aligned}$$

The Simplex Method

Problem. Maximize $4x_1 + x_2 - x_3 = z$

subject to
$$\begin{aligned} x_1 + \quad 3x_3 &\le 6 \\ 3x_1 + x_2 + 3x_3 &\le 9 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic Soln	
1	-4	-1	1	0	0	0		
0	1	0	3	1	0	6	(0,0,0,6,9)	$z=0$
0	3	1	3	0	1	9		
1	0	$1/3$	5	0	$4/3$	12		
0	0	$-1/3$	2	1	$-1/3$	3	(3,0,0,3,0)	$z=12$
0	1	$1/3$	1	0	$1/3$	3		

The Simplex Method

Problem. Maximize $z = x_1 + \frac{1}{2}x_2$

subject to

$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

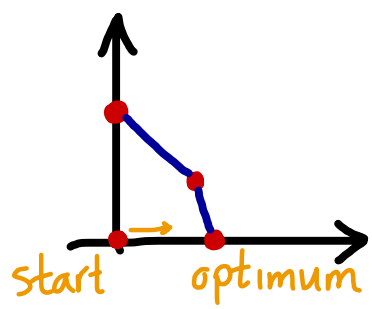
First we write the standard form:

$$\begin{aligned} z - x_1 - \tfrac{1}{2}x_2 \qquad &= 0 \\ 2x_1 + x_2 + x_3 \qquad &= 4 \\ x_1 + 2x_2 \quad + x_4 &= 3 \end{aligned}$$

The Simplex Method

$$
\begin{aligned}
z - x_1 - \tfrac{1}{2}x_2 \qquad &= 0 \\
2x_1 + x_2 + x_3 \qquad &= 4 \\
x_1 + 2x_2 \quad + x_4 &= 3
\end{aligned}
$$

z	x_1	x_2	x_3	x_4	RHS	Basic Soln
1	-1	-1/2	0	0	0	
0	2	1	1	0	4	$(0,0,4,3)$ $z=0$
0	1	2	0	1	3	
1	0	0	1/2	0	2	
0	1	1/2	1/2	0	2	$(2,0,0,1)$ $z=2$
0	0	3/2	-1/2	1	1	



Note: If you pivot on x_2 instead of x_1, it takes 3 steps instead of 1!

The Simplex Method

Example. Maximize $z = 3x + 4y$
subject to $2x + y \le 5$
$x, y \ge 0$

$z = 20$ at $(0,5)$

Example. Maximize $z = 3x + 2y$
subject to $x + y \le 4$
$2x + y \le 5$
$x, y \ge 0$

$z = 9$ at $(1,3)$

Example. Maximize $z = 3x + 2y$
subject to $2x + y \le 18$
$2x + 3y \le 42$
$3x + 2y \le 24$
$x, y \ge 0$

$z = 24$ at $(8,0)$

Example. Maximize $z = x_1 + 2x_2 - x_3$
subject to $2x_1 + x_2 + x_3 \le 14$
$4x_1 + 2x_2 + 3x_3 \le 28$
$2x_1 + 5x_2 + 5x_3 \le 30$
$x_i \ge 0$

$z = 13$ at $(5,4,0)$

The Simplex Method

Tableau for last example:

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS	Solution
1	-1	-2	1	0	0	0	0	
0	2	1	1	1	0	0	14	$(0,0,0)$ $z=0$
0	4	2	3	0	1	0	28	
0	2	5	5	0	0	1	30	
1	$-1/5$	0	3	0	0	$2/5$	12	
0	$8/5$	0	0	1	0	$-1/5$	8	$(0,6,0)$ $z=12$
0	$16/5$	0	1	0	1	$-2/5$	16	
0	$2/5$	1	1	0	0	$1/5$	6	
1	0	0	$49/16$	0	$1/16$	$3/8$	13	
0	0	0	$-1/2$	1	$-1/2$	0	0	$(5,4,0)$ $z=13$
0	1	0	$5/16$	0	$5/16$	$-1/8$	5	
0	0	1	$7/8$	0	$-1/8$	$1/4$	4	

The Simplex Method

Example. A company produces chairs and sofas.
A chair requires 3 hrs carpentry, 9 hrs finishing, 2 hrs upholstery.
A sofa requires 2 hrs carpentry, 4 hrs finishing, 10 hrs upholstery.
The company can afford 66 hours of carpentry, 180 hours of finishing, and 200 hours of upholstery.
The profit on a chair is \$90 and on a sofa is \$75.
How many chairs and sofas should be made to maximize profit?

Solution. Want to maximize $P = 90x_1 + 75x_2$

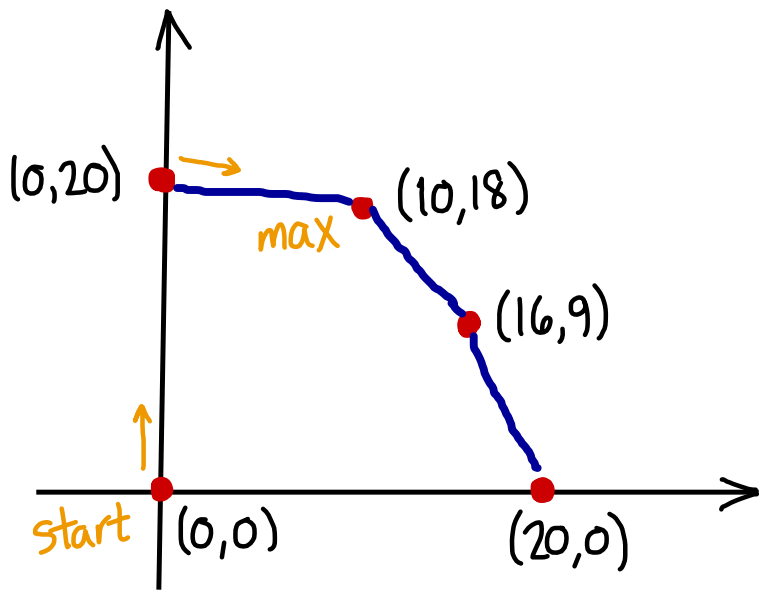
subject to
$$3x_1 + 2x_2 \leq 66$$
$$9x_1 + 4x_2 \leq 180$$
$$2x_1 + 10x_2 \leq 200$$

The Simplex Method

z	x_1	x_2	x_3	x_4	x_5	RHS	Solution
1	-90	-75	0	0	0	0	
0	3	2	1	0	0	66	(0,0) P=0
0	9	4	0	1	0	180	
0	2	10	0	0	1	200	
1	-75	0	0	0	$15/2$	1500	
0	$13/5$	0	1	0	$-1/5$	26	(0,20) P=1500
0	$41/5$	0	0	1	$-2/5$	100	
0	$1/5$	1	0	0	$1/10$	20	
1	0	0	$575/13$	0	$45/26$	2250	
0	1	0	$5/13$	0	$-1/13$	10	(10,18) P=2250
0	0	0	$-41/13$	1	$41/65$	18	
0	0	1	$-1/13$	0	$15/130$	18	

The Simplex Method

Picture for the last problem:



Geometry of the Simplex Method

We can write the constraints of a linear program as $Ax=b$.

The vertices of the feasible region are points x that lie in the feasible region and where the columns of A corresponding to the nonzero entries of x are linearly independent.

Two vertices span an edge if they are nonzero in all of the same coordinates except for 1.

Thus, swapping one basic variable for another corresponds to moving along an edge.

As we perform row operations, the feasible region (and objective function) changes, but the corresponding graph is naturally isomorphic to the original.

Review

1. A polygon is convex if the line segment connecting any two vertices of the polygon is completely contained in the polygon.

Prove that the sum of the interior angles of a convex n-gon is $(n-2)\pi$.

2. A finite collection of lines in the plane determines a "map." Show that this map can be colored with 2 colors.

3. Define a sequence a_n by $a_0 = 0$, $a_n = a_{\lfloor \frac{n}{5} \rfloor} + a_{\lfloor \frac{3n}{5} \rfloor} + n$

What is a_{10}?

Prove that $a_n \leq 20n$ for all n.

4. Let a_n denote the number of n-digit numbers, each of whose digits is 1, 2, 3, or 4 and in which the number of 1's is even.

What is a_3? Find a recursive formula for a_n.

5. Solve the recurrence relation $a_n = 4a_{n-1} + 8^n$, where $a_0 = 1$.

6. What is the generating function associated to the sequence $2, 6, 18, 54, \ldots$?

7. Use generating functions to solve $a_n = 3a_{n-1} + 1$, $a_0 = 1$.

8. How many ways are there to walk up a flight of n stairs if you can go up either 1 or 2 steps at a time?

9. True or false: If f is $O(g)$ and h is $O(g)$ then fh is $O(g)$.

10. True or false: $\log_2 n$ and $\log_3 n$ have the same order.

11. Say x is a real number and n a natural number. Here are two algorithms for computing x^{2^n}. Which is more efficient (in terms of number of multiplications)?

A. Set $a=1$
For $i=1$ to 2^n, replace
a by xa
Output a

B. Set $a=x$.
For $i=1$ to n, replace
a by a^2
Output a.

12. Show that $n \log n < n^2$.

13. Which of the following are equal?

A. The number of 2-element subsets of $\{1, \ldots, n\}$

B. The number of edges in K_n

C. $1 + 2 + \cdots + (n-1)$

D. The number of ways of putting 2 marbles in n boxes, any number to a box.

14. In a room with 19 cats, there are twice as many dumb cats as ugly cats. The number of cats that are neither dumb nor ugly is twice the number of cats that are dumb and ugly. There is one ugly cat who is not dumb. How many cats are dumb but not ugly?

15. Write down any list of 6 natural numbers. There must be a string of consecutive numbers with sum divisible by... ?

16. How many poker hands contain exactly 3 royal cards?

17. Ten students form 3 study groups, each with at least two people. How many ways are there to do this?

18. Bridge hands are dealt to Alice, Bob, Charley, and Daisy. What is the probability that Alice has 3 red cards given that she and her partner got 9 red cards?

19. For which n is K_n isomorphic to C_n?

20. For which m,n,k is $K_{m,n}$ isomorphic to C_k?

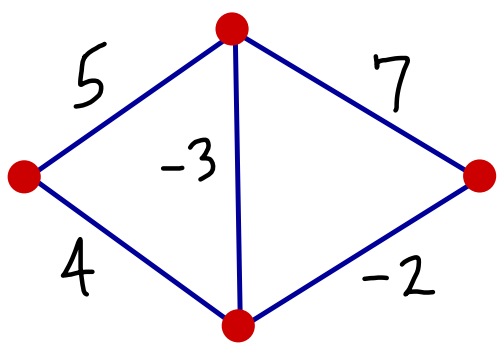
21. For which m,n is $K_{m,n}$ Eulerian? Hamiltonian?

22. Is it possible for a knight to visit every square of a chessboard and return to where it started?

What about a 7×7 board?

23. How many spanning trees does K_4 have? K_5?

24. Find the "distances" between all pairs of vertices using the Floyd-Warshall algorithm.

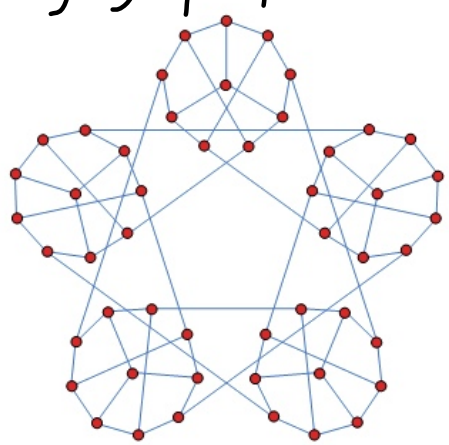


25. How many isomers of C_7H_{16} are there?

26. How many connected graphs have chromatic number 1?

27. The cone on a graph G is obtained by adding one new vertex and connecting it by edges to all old vertices. What is the chromatic number of the cone on $K_{3,2}$? What about the cone on C_n?

28. Is the following graph planar?



29. Solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$, $a_0 = 0$, $a_1 = 1$ using linear algebra.

30. Maximize $2x_1 + 4x_2 + 3x_3 + x_4$

subject to
$$3x_1 + x_2 + x_3 + 4x_4 \leq 12$$
$$x_1 - 3x_2 + 2x_3 + 3x_4 \leq 7$$
$$2x_1 + x_2 + 3x_3 - x_4 \leq 10$$
$$x_i \geq 0$$