Scores: 1 2 3 4 5 6 7 8 9 10 Name Prof. M

Mathematics 2602

Section L1
Midterm 1
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14 September 2011

1. State the principle of mathematical induction.

Suppose we have a mathematical statement that depends on an integer n. Suppose that

① The statement is true for $n = n_0$.

② Whenever the statement is true for $n = k$, it is true for $n = k+1$.

Then, the statement is true for all $n \geq n_0$.

2. State the definition of "f is $\mathcal{O}(g)$."

Let f and g be functions from $\mathbb{N}$ to $\mathbb{R}$.
We say f is $\mathcal{O}(g)$ if there is an integer n_0 and a positive real number c so that

$$|f(n)| \leq c|g(n)|$$

for all $n \geq n_0$.

3. Use induction to prove that $2^n + 3^n - 5^n$ is divisible by 6 for $n \geq 1$.

Base case: $n = 1$

$$2^1 + 3^1 - 5^1 = 0$$

0 divisible by 6 ✓

Assume the statement is true for $n = k$:

$2^k + 3^k - 5^k$ is divisible by 6.

Show that the statement is true for $n = k+1$.

$$2^{k+1} + 3^{k+1} - 5^{k+1}$$

$$= 2 \cdot 2^k + 3 \cdot 3^k - 5 \cdot 5^k$$

$$= 5 \cdot 2^k + 5 \cdot 3^k - 5 \cdot 5^k - 3 \cdot 2^k - 2 \cdot 3^k$$

$$= 5(2^k + 3^k - 5^k) - 6 \cdot 2^{k-1} - 6 \cdot 3^{k-1}$$

The first term is divisible by 6 by assumption. The second and third terms are clearly divisible by 6. Thus, the sum is divisible by 6.

By induction, $2^n + 3^n - 5^n$ is divisible by 6 for $n \geq 1$.

4. Let a_n be the number of ways to cover a $2 \times n$ checkerboard with dominoes (a domino is made of two squares glued along one edge). Use strong induction to show that a_n is equal to the $(n+1)$st Fibonacci number F_{n+1}. Recall that the Fibonacci numbers are given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Base cases: $n=0$

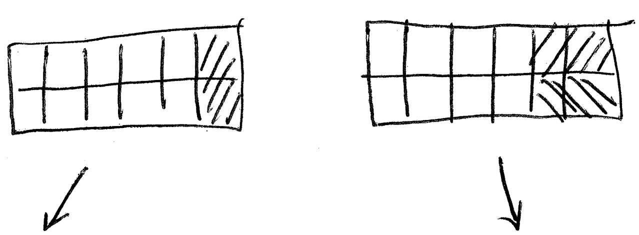
There is one way to cover a 2×0 board and $F_1 = 1$ ✓

$n=1$

Again, one way to cover

Assume the statement is true for $[0, k-1]$.

There are two ways to cover the end of a $2 \times k$ board:



By induction, can be completed in F_k ways

By induction, can be completed in F_{k-1} ways.

Thus the $2 \times k$ board can be covered in $F_k + F_{k-1} = F_{k+1}$ ways.

5. Solve the recurrence relation given by $a_0 = 0$, $a_1 = 30$, and

$$a_n = 10a_{n-1} - 25a_{n-2}$$

for $n \geq 2$.

Characteristic polynomial:

$$x^2 - 10x + 25 = 0$$

$$(x-5)^2 = 0$$

$$x = 5.$$

So $a_n = k_1 5^n + k_2 n 5^n$

$0 = k_1$

$30 = k_2 \cdot 5 \Rightarrow k_2 = 6$

So $a_n = 6n5^n$.

6. Solve the recurrence relation given by $a_0 = 2$ and

$$a_n = 3a_{n-1} - 4n$$

for $n \geq 1$.

First, find a particular solution p_n.

Since $4n$ linear, guess $p_n = mn + b$.

$$p_n = 3p_{n-1} - 4n$$

$$\begin{aligned} mn + b &= 3(m(n-1) + b) - 4n \\ &= 3(mn - m + b) - 4n \\ &= 3mn - 3m + 3b - 4n \\ &= (3m - 4)n + 3(b - m) \end{aligned}$$

So $3m - 4 = m$ and $3(b-m) = b$

$\rightsquigarrow m = 2$ $\qquad 3b - 3\cdot 2 = b$

$b = 3.$

$$p_n = 2n + 3$$

Then find a solution q_n to

$$a_n = 3a_{n-1}$$

So $q_n = k_1 3^n$.

Thus $a_n = k_1 3^n + (2n + 3)$

Solve for k_1:

$$2 = k_1 + 3 \rightsquigarrow k_1 = -1$$

$$a_n = -5 \cdot 3^n + 2n + 3$$

7. For each generating function, give the associate sequence. You do not need to show your work.

$\frac{7}{1+x}$ $\quad 7, -7, 7, -7 \quad a_n = (-1)^n 7$

$\left(\text{since } \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots\right)$

$\frac{x}{(1-x)^2}$ $\quad a_n = n$

$\left(\text{since } \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots\right)$

$\frac{1}{1-5x}$ $\quad a_n = 5^n.$

Use generating functions to solve the recurrence relation given by $a_0 = 2$ and $a_n = 5a_{n-1}$ for $n \geq 1$.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$-5x f(x) = -5a_0 x - 5a_1 x^2 - \cdots$$

$$(1-5x) f(x) = a_0 + 0 + 0 + \cdots$$

$$f(x) = \frac{2}{1-5x} \iff a_n = 2 \cdot 5^n$$

8. Use the definition of "f is $\mathcal{O}(g)$" in order to verify the following.

n^3 is not $\mathcal{O}(n^2)$

$$n^3 \le cn^2$$

means $n \le c$.

So it cannot be true that there is a c with $n^3 \le cn^2$ for all n large.

2^n is $\mathcal{O}(n!)$

Take $c=1$, $n_0=4$.

Want $2^n \le 1 \cdot n!$ for $n \geq 4$.

i.e. $\dfrac{2 \cdot 2 \cdot 2 \cdots 2}{n \cdot (n-1)(n-2) \cdots 1} \le 1$ for $n \geq 4$.

i.e. $\dfrac{2^{n-4}}{n(n-1) \cdots (n-3)} \cdot \dfrac{2^4}{4!} \le 1$.

Both terms are less than 1, so their product is also.

9. Find a function on the list

$$n^2, \quad 1, \quad n^3, \quad n\log n, \quad \log n, \quad n^4, \quad 3^n, \quad n^n, \quad n!, \quad n, \quad e^n, \quad n^5$$

that has the same order as each of the following functions. You do not need to show your work.

$10^{99}n^{452} + \dfrac{3^n}{1,000,000} + 15\log n$

3^n

$n\log n + n^2$

n^2

$e^n + n^e$

e^n

$3n! - 2^n$

$n!$

$n + \log n$

n

10. Let A and B be positive real numbers. Show that $\log(An + B)$ and $\log n$ have the same order.

$$\lim_{n \to \infty} \frac{\log(An+B)}{\log n} \overset{L'H}{=} \lim_{n\to\infty} \frac{A/An+B}{1/n}$$

$$= \lim_{n\to\infty} An / An + B = 1 = \text{constant}$$

Thus $\log(An+B) \asymp \log n$.

Let $f : \mathbb{N} \to \mathbb{R}$ be a function with $\lim_{n\to\infty} f(n) = \infty$. Let M and N be nonzero real numbers. Show that the function $Mf(n) + N$ has the same order as $f(n)$.

$$\lim_{n\to\infty} \frac{Mf(n) + N}{f(n)} = \lim_{n\to\infty} \frac{M + N/f(n)}{1}$$

$$= M = \text{constant}.$$

Thus $Mf(n) + N \asymp f(n)$.