Scores: 1–4 5 6 7 8 9 10 Name ______________________

Section L___

Mathematics 2602

Midterm 1

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1. Answer the following questions with your clicker, and record your answers on this page.

$\boxed{\text{Clicker \#1}}$ Choose the sentence that best completes the statement of the principle of mathematical induction.

Say we have a mathematical statement that depends on an integer n. Suppose:

- The statement is true for $n = n_0$.
- ______________________________.

Then the statement is true for all $n \geq n_0$.

A. The statement is true for all $n = k$.

B. Whenever the statement is true for $n = k$, it is also true for $n = k + 1$.

C. If the statement is true for $n = k$, then it is true for all $n \geq n_0$.

D. If the statement is true for $n = n_0$, then it is also true for all $n = k$.

$\boxed{\text{Clicker \#2}}$ Choose the phrase that best completes the definition of "f is $\mathcal{O}(g)$."

Let f and g be functions $\mathbb{N} \to [0, \infty)$. We say that f is $\mathcal{O}(g)$ if ______________ so that

$$f(n) \leq cg(n)$$

for all $n \geq n_0$.

A. for all natural numbers n_0, there exists a positive real number c

B. for all positive real numbers c, there exists a natural number n_0

C. for all natural numbers n_0 and all positive real numbers c

D. there exists a natural number n_0 and a positive real number c

2. Answer the following questions with your clicker, and record your answers on this page.

$\boxed{\text{Clicker \#3}}$ True or false: generating functions are useless because you can always solve the problem in a simpler way.

$\boxed{\text{Clicker \#4}}$ Recall that the Fibonacci numbers are defined by the recursion relation

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1.$$

What is F_5?

$F_1 = 1 + 0 = 1$

$F_2 = 1 + 1 = 2$

$F_3 = 2 + 1 = 3$

$F_4 = 3 + 2 = 5$

$\boxed{F_5 = 5 + 3 = 8}$

$\boxed{\text{Clicker \#5}}$ Which of the following are linear recurrence relations? Select all that apply.

A. $a_n = a_{n-2}$

B. $a_n = -3a_{n-1} + 15a_{n-2}$

C. $a_n = a_{n-1}a_{n-2}$

D. $a_n = a_n^2 + a_{n-1}$

$\boxed{\text{Clicker \#6}}$ Put the following orders of complexity in order, from smallest to largest.

A. $\mathcal{O}(n^{500,000})$

B. $\mathcal{O}(2^n)$

C. $\mathcal{O}(1)$

D. $\mathcal{O}(\log n)$

$1 \prec \log n \prec n^{500,000} \prec 2^n$

So C, D, A, B

$\boxed{\text{Clicker \#7}}$ Discrete is the opposite of...

A. *continuous*

B. *finite*

C. *infinite*

D. *pure*

Answer the following questions with your clicker, and record your answers on this page.

3. Find a function on the list below that has the same order as each of the following functions.

Clicker #8	$10^{99}n^{452} + \frac{3^n}{1,000,000} + 15\log n$	$\boxed{G: 3^n}$
Clicker #9	$\cancel{e^n n^e}$ $e^n + n^e$	$\boxed{J: e^n}$
Clicker #10	$3n! + 2^n$	$\boxed{I: n!}$

A.	n^2	F.	n^e
B.	n^{452}	G.	3^n
C.	2^n	H.	n^n
D.	$n \log n$	I.	$n!$
E.	$\log n$	J.	e^n

4. Match each generating function to one of the sequences listed below.

Clicker #11	$\frac{7}{1+x} = 7 - 7x + 7x^2 - 7x^3 + \ldots$	$\boxed{D: (-1)^n 7}$
Clicker #12	$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + \ldots$	$\boxed{B: n}$
Clicker #13	$\frac{1}{1-5x} = 1 + 5x + 5^2x^2 + 5^3x^3 + \ldots$	$\boxed{C: 5^n}$

A.	1	F.	$n-1$
B.	n	G.	$n+2$
C.	5^n	H.	7^n
D.	$(-1)^n 7$	I.	7
E.	$n+1$	J.	$(-1)^n 5$

5. Use induction to prove that $n^3 + 2n$ is divisible by 3 for $n \geq 0$.

Base Case:

For $n=0$, $n^3+2n=0$ which is divisible by 3

Induction Hypothesis:

Assume k^3+2k is divisible by 3 for some $k \geq 0$.

Induction Step:

$$(k+1)^3 + 2(k+1) = k^3+3k^2+3k+1+2k+2$$
$$= k^3+2k+3(k^2+k+1).$$

k^3+2k is divisible by 3 by induction hypothesis

$3(k^2+k+1)$ is clearly a multiple of 3

therefore $(k+1)^3+2(k+1)$ is divisible by 3.

By the principle of mathematical induction,

n^3+2n is divisible by 3 for all $n \geq 0$.

6. Recall that the Fibonacci numbers are defined by the recursion relation

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1.$$

Use the strong version of the principle of mathematical induction to show that, for $n \geq 1$, the number of n-digit binary strings with no consecutive 1's is F_{n+2}.

As one example, 10001001000001 is a 14-digit binary string with no consecutive 1's.

Let a_n be the number of n-digit strings with no consecutive 1's.
We want to prove $a_n = F_{n+2}$ for all $n \geq 1$. Note $F_2 = 1$, $F_3 = 2$, $F_4 = 3$

Base Cases:

For $n=1$, the length 1 strings are 0 and 1,
so $a_1 = 2 = F_3$.

For $n=2$, the length 2 strings are 00, 01 and 10,
so $a_2 = 3 = F_4$.

Induction Hypothesis:

Assume $a_\ell = F_{\ell+2}$ for $1 \leq \ell < k$ for some $k \geq 1$.

Induction Step:

A k-digit string can begin either with 0 or 1

Case 1: If the first digit is 0, the remaining digits can be any (k-1)-digit string with no repeated 1's.
There are a_{k-1} ways for this to happen.

Case 2: If the first digit is 1, the second digit must be 0. After that, the remaining digits can be and (k-2)-digit string with no repeated 1's.
There are a_{k-2} ways for this to happen.

So in total $a_n = a_{k-1} + a_{k-2}$.

By the induction hypothesis, $a_{k-1} + a_{k-2} = F_{k+1} + F_k = F_{k+2}$

so $a_k = F_{k+2}$.

By the principle of mathematical induction, $a_n = F_{n+2}$ for all $n \geq 1$.

7. Consider the recurrence relation given by $a_0 = 2$ and

$$a_n = 5a_{n-1}$$

for $n \geq 1$. Solve for a_n using generating functions.

$$f(x) = a_0 + \cancel{a_1 x} + \cancel{a_2 x^2} + \cancel{\cdots} \cancel{+ a_n x^n} + \cdots$$

$$-5xf(x) = \cancel{-5a_0 x} - \cancel{5a_1 x^2} - \cancel{\cdots} - \cancel{5a_{n-1}x^n} - \cdots$$

$$f(x) - 5xf(x) = a_0 + 0x + 0x^2 + \cdots + 0x^n + \cdots$$

using $a_0 = 2$,

$$f(x)(1-5x) = 2,$$

$$f(x) = \frac{2}{1-5x} = 2 \cdot 1 + 2 \cdot 5x + 2 \cdot 5^2 x^2 + \cdots$$

The corresponding sequence is

$$a_n = 2 \cdot 5^n$$

8. Solve the recurrence relation given by $a_0 = 2$, $a_1 = 0$, and

$$a_n = 2a_{n-1} + 3a_{n-2} + 4$$

for $n \geq 2$.

$$a_n = q_n + p_n$$

characteristic polynomial:

$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$r_1 = 3, \quad r_2 = -1$$
$$q_n = c_1 3^n + c_2(-1)^n$$

particular solution:

Non-homogeneous part is a constant so let $p_n = b$

$$p_n = 2p_{n-1} + 3p_{n-2} + 4$$
$$b = 2b + 3b + 4$$
$$4b = -4$$
$$b = -1$$
$$p_n = -1$$

Combine the parts of the solution:

$$a_n = q_n + p_n = c_1 3^n + c_2(-1)^n - 1$$
$$a_0 = c_1 3^0 + c_2(-1)^0 - 1 = 2$$
$$a_1 = c_1 3^1 + c_2(-1)^1 - 1 = 0$$
$$c_1 + c_2 = 3$$
$$3c_1 - c_2 = 1$$
$$3c_1 - (3 - c_1) = 1$$
$$c_1 = 1$$
$$c_2 = 2$$

$$\boxed{a_n = 3^n + 2(-1)^n - 1}$$

9. Use the definition of "f is $\mathcal{O}(g)$" in order to verify that

$$n^n \neq \mathcal{O}(n!).$$

If it were the case that $n^n = \mathcal{O}(n!)$ then there is some $c > 0$ such that

$$n^n \leq cn!$$

for all $n \geq n_0$ for some n_0, so $\frac{n^n}{n!} \leq c$.

Note $\frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} = n \cdot \frac{n}{2} \cdot \frac{n}{3} \cdots \frac{n}{n}$.

Each term $\frac{n}{2}, \frac{n}{3}, \dots, \frac{n}{n}$ is greater than or equal to 1.

so the product $n \cdot \frac{n}{2} \cdots \frac{n}{n}$ is greater than or equal to n

For any $c > 0$, choose $n > c$.

Then $\frac{n^n}{n!} \geq n > c$, so $n^n \neq \mathcal{O}(n!)$

10. Show that $n \ln n \prec n^2$.

$$\lim_{n\to\infty} \frac{n\ln(n)}{n^2} = \lim_{n\to\infty} \frac{\ln(n)}{n} \overset{\text{L'Hopital's Rule}}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$$

so $n\ln(n) \prec n^2$.

Let A and B be natural numbers with $A > B$. Show that $A^n + B^n \asymp A^n$.

$$\lim_{n\to\infty} \frac{A^n+B^n}{A^n} = \lim_{n\to\infty} \frac{A^n}{A^n} + \frac{B^n}{A^n} = \lim_{n\to\infty} 1+\left(\frac{B}{A}\right)^n = 1 + \lim_{n\to\infty}\left(\frac{B}{A}\right)^n = 1+0$$

since $\left|\frac{B}{A}\right| < 1$, this limit is 0

1 is greater than 0 and less than ∞

so $A^n+B^n \asymp A^n$