Scores: 1 2 3 4 5 6 7 8 9 10 Name Prof. M

Mathematics 2602

Section L1
Midterm 2
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1. State the strong form of the pigeonhole principle.

If n objects are placed in m boxes then one box must contain at least $\lceil \frac{n}{m} \rceil$ objects.

State the binomial theorem.

For any x, y and any $n \in \mathbb{N}$:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

2. Answer each question using one of the following:

$$26^3, \quad 3^{26}, \quad 26!, \quad \binom{28}{3}, \quad 26 \cdot 3, \quad P(26,3), \quad D_{26}, \quad \binom{26}{3}, \quad \binom{3}{26}.$$

How many ways are there are of choosing three letters of the alphabet if repeated letters are allowed?

$\binom{28}{3}$

How many ways are there are of choosing three letters of the alphabet if repeated letters are not allowed?

$\binom{26}{3}$

How many ways are there of making a string of three letters if repeated letters are allowed?

26^3

How many ways are there of making a string of three letters if repeated letters are not allowed?

$P(26,3)$

How many ways are there of arranging the letters of the alphabet into a string so that no letter is in its usual position?

D_{26}

3. How many even three-digit numbers have no repeated digits?

Case 1: Last digit 0

9 choices for first digit

8 for second

72.

Case 2: Last digit not 0.

4 choices for last digit

8 choices for first

8 choices for second.

256

$256 + 72 = 328$

5. Suppose that a student government has 10 representatives, and that there are 3 committees with 5 representatives each and 2 committees with 3 representatives each. Is it necessarily the case that some representative is on 3 committees? Explain your answer.

objects : committee assigments

21

boxes : representatives

10

By pigeon hole principle

Some rep has

$$\left\lceil \frac{21}{10} \right\rceil = 3$$

committees.

4. How many integers between 1 and 100 are multiples of either 3, 7, or 11?

$$A_k = \{n : 1 \leq n \leq 100,\ n \text{ divis. by } k\}$$

$$|A_3 \cup A_7 \cup A_{11}| = |A_3| + |A_7| + |A_{11}|$$
$$- |A_3 \cap A_7| - |A_3 \cap A_{11}| - |A_7 \cap A_{11}|$$
$$+ |A_3 \cap A_7 \cap A_{11}|$$

$$= |A_3| + |A_7| + |A_{11}|$$
$$- |A_{21}| - |A_{33}| - |A_{77}|$$
$$+ |A_{231}|$$

$$= \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{7} \right\rfloor + \left\lfloor \frac{100}{11} \right\rfloor$$
$$- \left\lfloor \frac{100}{21} \right\rfloor - \left\lfloor \frac{100}{33} \right\rfloor - \left\lfloor \frac{100}{77} \right\rfloor$$
$$+ \left\lfloor \frac{100}{231} \right\rfloor$$

$$= 33 + 14 + 9 - 4 - 3 - 1 + 0$$

$$= 48$$

6. You roll two dice. The first one is fair and the second one has probabilities $P(1) = 1/3$, $P(2) = 1/6$, $P(3) = P(4) = P(5) = P(6) = 1/8$. What is the probability that the sum of the two rolls is 5?

fair unfair

$$P(1,4) = \frac{1}{6} \cdot \frac{1}{8}$$

$$P(2,3) = \frac{1}{6} \cdot \frac{1}{8}$$

$$P(3,2) = \frac{1}{6} \cdot \frac{1}{6}$$

$$P(4,1) = \frac{1}{6} \cdot \frac{1}{3}$$

$$\frac{1}{6}\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{6} + \frac{1}{3}\right).$$

7. There are seven knights: Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, and Saturday.

In how many ways is it possible for the seven knights to sit at a round table if Saturday and Sunday insist on sitting next to Friday?

Friday sits somewhere.

7 choices.

Sat sits next to Fri

2 choices

Sun sits next to Fri

1 choice.

Other 4 knights sit anywhere

4! choices.

The 7 rotations of the table give same configuration.

$$\frac{7 \cdot 2 \cdot 1 \cdot 4!}{7} = 48$$

8. Find the coefficient of x^{12} in $\left(\frac{3}{x}+x^2\right)^9$.

$$\left(\frac{3}{x}+x^2\right)^9 = \sum_{k=0}^{9}\binom{9}{k}\left(\frac{3}{x}\right)^k\left(x^2\right)^{9-k}$$

$$= \sum_{k=0}^{9}\binom{9}{k}3^k\left(\frac{1}{x}\right)^k x^{18-2k}$$

k^{th} coefficient: $\binom{9}{k}3^k$

k^{th} power of x: $18-3k$

want $18-3k=12$

$k=2.$

$\binom{9}{2}\cdot 3^2$

9. You are putting 5 identical red marbles, 10 identical white marbles, and 15 identical blue marbles into 12 boxes. You can put as many marbles as you want into each box. How many ways are there to do this, if you need to put at least one of each color in the first box?

First put red.

1 choice for first marble.

$\binom{12+4-1}{4}$ choices for other 4.

Then put white.

$\binom{20}{9}$

Then blue

$\binom{25}{14}$

$$\binom{15}{4}\binom{20}{9}\binom{25}{14}$$

10 (Extra credit). I have two coins. The first one is fair and the second one is unfair and comes up heads 3/4 of the time. I pick one of the two coins randomly (there is a 1/2 chance that the fair coin is chosen and a 1/2 chance the unfair coin is chosen) and hand it to you.

You flip the coin 4 times. What is the probability of getting 4 tails?

$$P(TTTT) = P(TTTT/F)\,P(F) + P(TTTT|U)\,P(U)$$

$$= \left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)^4\left(\frac{1}{2}\right)$$

$$= \frac{1}{32} + \frac{1}{512}$$

$$= \frac{17}{512}.$$

You flip 4 times and get 4 tails. What is the probability that the fair coin was chosen?

$$P(F|TTTT) = \frac{P(TTTT|F)\,P(F)}{P(TTTT)}$$

$$= \frac{\frac{1}{16}\cdot\frac{1}{2}}{17/512}$$

$$= 16/17.$$