Scores: 1–4 5 6 7 8 9 10 E

Name Solution

Section L___

Mathematics 2602

Midterm 2

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1. Answer the following questions with your clicker, and record your answers on this page.

$\boxed{\text{Clicker \#1}}$ Choose the phrase that correctly completes the statement of the strong form of the pigeonhole principle.

If we put m objects into n boxes, then...

A. all of the boxes must have at least $\lceil \frac{m}{n} \rceil$ *objects in them.*

B. at least $\lceil \frac{m}{n} \rceil$ *boxes must have at least one object in them.*

C. there must be at least $\lceil \frac{m}{n} \rceil$ *boxes.*

D. there must be one box with at least $\lceil \frac{m}{n} \rceil$ *objects in it.*

D

$\boxed{\text{Clicker \#2}}$ Choose the statement that best describes the multiplication rule.

A. The number of ways in which a sequence of independent events can occur is the product of the number of ways in which each event can occur.

B. The number of ways in which a sequence of dependent events can occur is the product of the number of ways in which each event can occur.

C. The number of ways in which a sequence of dependent events can occur is the sum of the number of ways in which each event can occur.

D. The number of ways in which a sequence of independent events can occur is the sum of the number of ways in which each event can occur.

A

2. Answer the following questions with your clicker, and record your answers on this page.

$\boxed{\text{Clicker \#3}}$ How many people would you neeed to gather to ensure that there are two people with the same first initial?

27

$\boxed{\text{Clicker \#4}}$ True or false: In a given experiment, it is possible for two different events to each have probability 2/3.

Yes.

Roll a die. Let A be the event that you get 1, 2, 3, 4.

Let B be the event that you get 1, 2, 3, 5.

$\boxed{\text{Clicker \#5}}$ In how many ways can we rearrange the letters of the word PUPPY?

$$\frac{5!}{3!} = 20$$

$\boxed{\text{Clicker \#6}}$ I choose two numbers from $\{1, \ldots, 10\}$, both randomly. What is the probability that the sum is 18?

The first number should be at least $18 - 10 = 8$.

And the second number is determined by the first number if you want the sum is 18.

So the probability is $\frac{3}{10} \cdot \frac{1}{10} = 0.03$.

$\boxed{\text{Clicker \#7}}$ How many derangements are there of the set $\{a, b, c\}$?

$$D_3 = 3! - 3 \cdot 2! + 3 \cdot 1! - 0!$$
$$= 2$$

3. Answer the following questions with your clicker, and record your answers on this page.

$\boxed{\text{Clicker \#8}}$ In a group of kids, 5 kids like red, 7 kids like blue, 4 kids like both, and 2 kids do not like blue. How many kids are there?

$5-4=1$ kid like red but doesn't like blue.

$2-1=1$ kid doesn't like red and blue.

So total number of kids $= 7+1+1=9$.

$\boxed{\text{Clicker \#9}}$ How many ways are there to arrange 5 people at a round table?

$$\frac{5!}{5} = 24$$

$\boxed{\text{Clicker \#10}}$ True or false: Alice and Bob each flip five coins. The events "Alice got 5 tails" and "Alice got more tails than Bob" are independent.

True.

$\boxed{\text{Clicker \#11}}$ True or false: Alice and Bob each flip five coins. The events "Alice got 5 tails" and "Alice got more tails than Bob" are mutually exclusive.

False.

$\boxed{\text{Clicker \#12}}$ The number of derangments of $\{1, \ldots, n\}$ is approximately...

A. e

B. $1/e$

C. $n!/e$

D. $1/e^n$

C

$$D_n = \sum_{k=0}^{n} n!(-1)^k \frac{1}{k!} \approx n! \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} = \frac{n!}{e}.$$

4. Answer the following questions with your clicker by matching each phrase to the corresponding formula and record your answers on this page.

$\boxed{\text{Clicker \#13}}$ The number of ways n people can sit on a bus with n seats.

$n!$

$\boxed{\text{Clicker \#14}}$ The number of ways r people can sit on a bus with n seats.

$P(n,r)$

$\boxed{\text{Clicker \#15}}$ The number of ways of n people can sit on a bus with n seats so that nobody sits in their assigned seat.

D_n

$\boxed{\text{Clicker \#16}}$ The number of ways of putting r people into n buses, if we do not care where people sit (each bus can hold as many people as you want).

n^r

$\boxed{\text{Clicker \#17}}$ The number of ways to choose which of the n people waiting for the bus gets a ride, if there are r seats available, and we fill every seat.

$\binom{n}{r}$

$A.$	$P(n,r)$	$F.$	$\binom{n}{r}$
$B.$	n^r	$G.$	$\binom{n+r-1}{r}$
$C.$	r^n	$H.$	$\binom{n-r+1}{n}$
$D.$	rn	$I.$	$n!$
$E.$	$\binom{r}{n}$	$J.$	D_n

5. How many integers between 1 and 1,000,000 are neither perfect squares nor perfect cubes?

Let A be the set of perfect squares between 1 and 1000000.

Let B be the set of perfect cubes between 1 and 1000000.

Since $1000000 = 10^6$, $|A| = 10^3$ and $|B| = 10^2$, and $|A \cap B| = 10$.

So the number of integers between 1 and 1000000 are neither perfect squares nor perfect cubes is $|\{1, 2, \cdots, 1000000\}| - |A| - |B| + |A \cap B|$

$= 10^6 - 10^3 - 10^2 + 10$

$= 998910.$

6. Suppose you are given a collection of 13 natural numbers. Explain why there must be two numbers in the collection whose difference is divisible by 10.

There are only 10 possible numbers for the last digit, so there are 2 numbers among the 13 numbers in the collection that have the same last digit.

Then the difference of these 2 numbers has 0 as the last digit, so it is divisible by 10.

7. You want to make a bag of 82 jelly beans. There are 20 flavors available. There are large supplies of most flavors, but there are only 81 cherry-flavored jelly beans and only 2 coconut-flavored jelly beans. In how many ways can you make your bag of jelly beans?

The number of ways to make my bag of jelly beans with exactly i coconut-flavored jelly beans

$$= \binom{82+19-i-1}{18} = \binom{100-i}{18}.$$

Because we want at most 2 coconut-flavored jelly beans but not all of them are cherry-flavored, the number of ways we can do is

$$\binom{100}{18} + \binom{99}{18} + \binom{98}{18} - 1$$

#.

8. From a standard deck of 52 cards, you deal 13 cards each to Alice, Bob, Charley, and Daisy. What is the probability that Alice gets 3 hearts, given that Alice and Bob together got 11 red cards?

Recall that a deck of cards has 13 each of hearts, diamonds, spades, and clubs. Hearts and diamonds are red, spades and clubs are black.

P(Alice gets 3 hearts | Alice and Bob together got 11 red cards)

$$= \frac{P(\text{Alice gets 3 hearts and Alice and Bob together got 11 red cards})}{P(\text{Alice and Bob together got 11 red cards})}$$

$$= \frac{\dfrac{\sum_{k=3}^{13}\binom{13}{k}\binom{13}{11-k}\binom{k}{3}\binom{26}{15}\binom{15+11-k}{13-k}}{\binom{52}{13}\binom{39}{13}}}{\dfrac{\binom{26}{11}\binom{26}{15}\binom{26}{13}}{\binom{52}{13}\binom{39}{13}}}$$

$$= \frac{\sum_{k=3}^{13}\binom{13}{k}\binom{13}{11-k}\binom{k}{3}\binom{26-k}{13-k}}{\binom{26}{11}\binom{26}{13}}$$

If we change the condition to be that "Alice and Bob together got 3 red cards," then the probability is

$$\frac{\binom{13}{3}\binom{26}{23}\binom{23}{10}}{\binom{26}{3}\binom{26}{23}\binom{26}{13}} = \frac{\binom{13}{3}\binom{23}{10}}{\binom{26}{3}\binom{26}{13}}.$$

9. I have three coins. One is fair, one comes up heads 1/3 of the time, and one comes up heads 1/4 of the time. I choose one coin at random, flip it, and get tails. What is the probability I chose the fair coin?

$$P(\text{the coin is fair} \mid \text{get tails})$$

$$= \frac{P(\text{choose the fair coin and get tails})}{P(\text{get tails})}$$

$$= \frac{\frac{1}{2}}{\frac{1}{3}\left[\frac{1}{2} + (1-\frac{1}{3}) + (1-\frac{1}{4})\right]}$$

$$= \frac{18}{23}.$$

10. What is the coefficient of x in the expansion of

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^{10}?$$

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^{10} = \sum_{k=0}^{10} \binom{10}{k} (\sqrt{x})^{10-k} \left(\frac{-1}{\sqrt{x}}\right)^{k}$$

$$= \sum_{k=0}^{10} \binom{10}{k} (-1)^k \, x^{5-k}.$$

So the coefficient of x is $\binom{10}{4}(-1)^4$

$$= \binom{10}{4}.$$