Scores: 1 2 3 4 5 6 7 8 9 10 E   Name KEY

Section L___

Mathematics 2602

Final Exam
Prof. Margalit
3 May 2012

1. Answer the following questions with your clicker, and record your answers on this page.

Clicker #1 Which describes an eigenvector v with eigenvalue λ for a linear map T?

A. $T(\lambda) = \lambda v$

B. $T(\lambda v) = v$

C. $T(v) = \lambda v$

D. $T(\lambda v) = \lambda v$

Clicker #2 Which statement best describes the idea of the simplex method?

A. If you move from corner to corner in the feasible region, always increasing the objective function, then you will find the optimum.

B. If you compute the value of the objective function at every corner of the feasible region, you will find the optimum.

C. If you watch the plane $z = N$ pass through the feasible region as N increases from $-\infty$ to ∞, the last point of the feasible region the plane hits will be a corner.

D. The optimum is obtained by reducing all slack variables to zero.

$\boxed{\text{Clicker \#3}}$ The third matrix found in an application of the Floyd–Warshall algorithm is:

$$M_2 = \begin{pmatrix} 0 & \infty & 2 \\ \infty & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

What is distance between vertices 1 and 2?

3

$\boxed{\text{Clicker \#4}}$ What is the $n = 3$ term of the sequence associated to the generating function

$$\frac{3x}{(1-x)^2}?$$

9

$\boxed{\text{Clicker \#5}}$ In how many ways can we choose positive integers a, b, c, and d so that

$$a + b + c + d = 11?$$

120

2. Recall that the Fibonacci numbers are defined by the recursion relation

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1.$$

Use the principle of mathematical induction to show that

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}$$

for all $n \geq 1$.

Base Case $(n=1)$:

$F_1 = 1$

$F_{2n} = F_2 = F_1 + F_0 = 1 + 0 = 1.$

$1 = 1.$

Induction Hypothesis:

Assume $F_1 + \ldots + F_{2k-1} = F_{2k}$ for some $k \geq 1$.

Induction Step:

$$F_1 + \ldots + F_{2(k+1)-1} = \underbrace{F_1 + \ldots + F_{2k-1}}_{\shortparallel} + F_{2k+1}$$

$F_{2k} + F_{2k+1}$ by induction hypothesis

$F_{2k} + F_{2k+1} = F_{2(k+1)}$ by the definition of the Fibonacci numbers.

so $F_1 + \ldots + F_{2(k+1)-1} = F_{2(k+1)}$

Therefore $F_1 + \ldots + F_{2n-1} = F_{2n}$ for all $n \geq 1$ by the principle of mathematical induction.

3. Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 3n$$

with initial conditions $a_0 = 2$ and $a_1 = 11$.

$$\boxed{a_n = -14\cdot 2^n + \frac{43}{4}\cdot 3^n + \frac{3}{2}n + \frac{21}{4}}$$

Homogeneous Part

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x = 2, 3$

$q_n = c_1 2^n + c_2 3^n$

Particular Solution

$p_n = an + b$

$an + b = 5(a(n-1)+b) - 6(a(n-2)+b) + 3n$

n terms: $an = 5an - 6an + 3n$

$2an = 3n$

$a = \frac{3}{2}$

constant terms: $b = -5a + 5b + 12a - 6b$

$b = -b + \frac{21}{2}$

$b = \frac{21}{4}$ $\qquad p_n = \frac{3}{2}n + \frac{21}{4}$

$a_n = q_n + p_n = c_1 2^n + c_2 3^n + \frac{3}{2}n + \frac{21}{4}$

$a_0 = c_1 + c_2 + \frac{21}{4} = 2$

$a_1 = 2c_1 + 3c_2 + \frac{3}{2} + \frac{21}{4} = 11$

$c_1 = -\frac{13}{4} - c_2$

$2(-\frac{13}{4} - c_2) + 3c_2 + \frac{27}{4} = 11$

$-\frac{26}{4} + c_2 + \frac{27}{4} = \frac{44}{4}$

$c_2 = \frac{43}{4}$

$c_1 = -\frac{13}{4} - \frac{43}{4} = -\frac{56}{4} = -14$

Check: $a_0 = -14 + \frac{43}{4} + \frac{21}{4} = -\frac{56}{4} + \frac{64}{4} = 2$ ✓

$a_1 = -28 + \frac{129}{4} + \frac{6}{4} + \frac{21}{4}$

$= -28 + \frac{156}{4} = -28 + 39 = 11$ ✓

$a_2 = -56 + \frac{387}{4} + 3 + \frac{21}{4}$

$= -53 + \frac{408}{4} = -53 + 102$

$= 49$ ✓

$a_2 = 5\cdot 11 - 6\cdot 2 + 3\cdot 2$

$= 55 - 12 + 6 = 49$

4. Use the definition of "f is $\mathcal{O}(g)$" to show that $n^2 + n$ is $\mathcal{O}(n^2)$.

choose $c = 2$, $n_0 = 1$

WTS: $n^2 + n \leq 2n^2$ for all $n \geq 1$.

~~since~~

since $1 \leq n$,

$n \cdot 1 \leq n \cdot n$

so $n^2 + n \cdot 1 \leq n^2 + n \cdot n$

$n^2 + n \leq 2n^2$

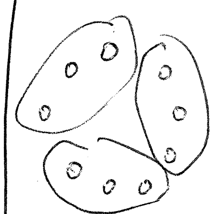
for all $n \geq n_0 = 1$

Therefore $n^2 + n$ is $\mathcal{O}(n^2)$.

Suppose you arrange the numbers 1 though 9 around a circle. Explain why there must be three consecutive numbers whose sum is at least 12.

Proof #1:

Look at the position with the number 9.
The minimum possible values of the neighboring numbers are 1 and 2.
Then the sum of these numbers is $9 + 1 + 2 = 12$.
Therefore 9 and its neighbors will always sum to ≥ 12.



Proof #2:

group the numbers into 3 sets of 3 consecutive numbers.
Suppose none of the sets sum to 12 or more. Then the total of all three sets is $< 3 \cdot 12 = 36$.
However the sum $1 + \ldots + 9 = 45$
so this is a contradiction.
One of the sets must sum to at least 12.

5. From a standard deck of 52 cards, you deal 13 cards each to Alice, Bob, Charley, and Daisy. What is the probability that Alice gets 3 hearts, given that Alice and Bob together get 3 hearts?

$$\mathbb{P}(\text{Alice gets 3 hearts} \mid \text{Alice and Bob together get 3 hearts})$$

$$= \frac{\binom{13}{3}\binom{39}{10}\binom{29}{13}}{\binom{13}{3}\binom{39}{23}\binom{26}{13}}$$

$$= \frac{\binom{39}{10}\binom{29}{13}}{\binom{39}{23}\binom{26}{13}}$$

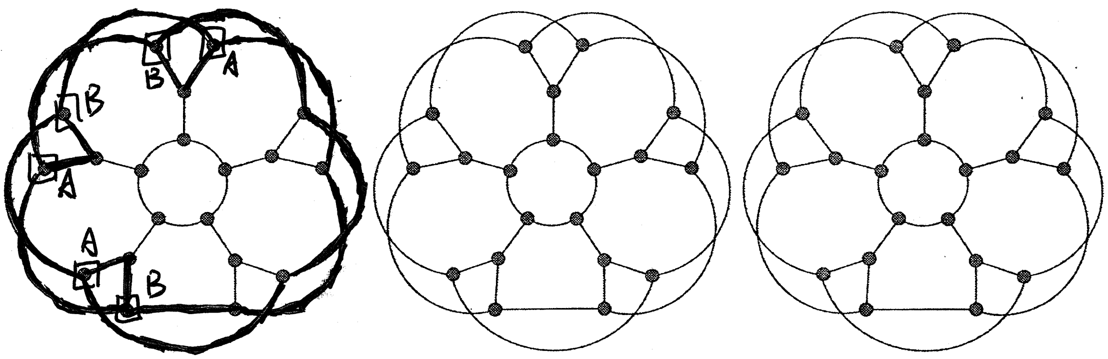
Expand and simplify the expression $\left(x + \frac{2}{x}\right)^5$.

$$\left(x+\frac{2}{x}\right)^5 = \sum_{k=0}^{5} \binom{5}{k} x^k \left(\frac{2}{x}\right)^{5-k}$$

$$= \sum_{k=0}^{5} \binom{5}{k} 2^{5-k} x^{2k-5}$$

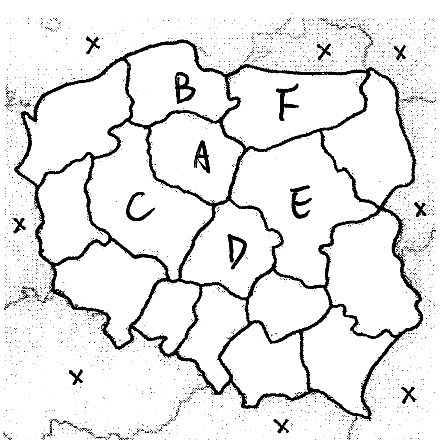
$$= 32\frac{1}{x^5} + 80\frac{1}{x^3} + 80\frac{1}{x} + 40x + 10x^3 + x^5$$

6. Explain why the following graph is not planar (there are three copies for your convenience).



The subgraph obtained by shaded edges is homeomorphic to $K_{3,3}$, so it is not planar.

What is the chromatic number of the following map of Poland? Explain your answer.



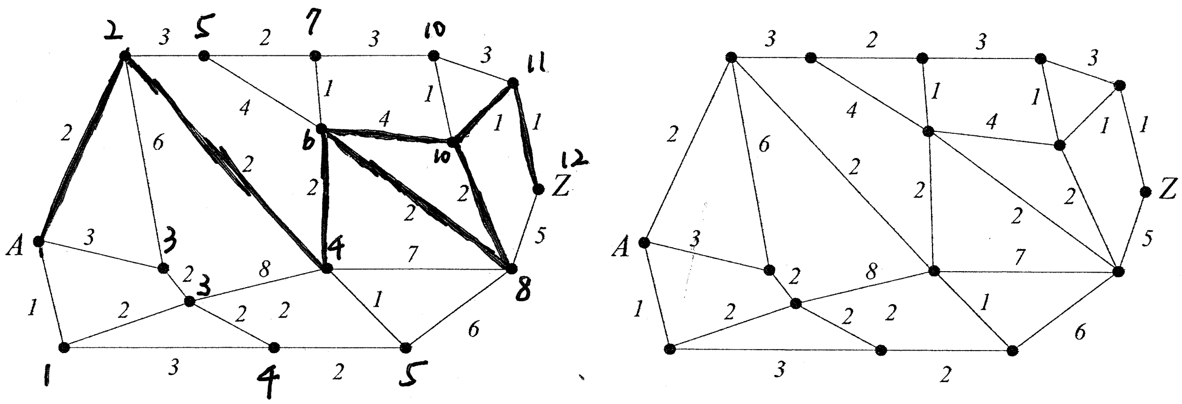
By the Four Color Theorem, the map is 4-colorable, So the chromatic number is at most 4.

And vertices B, C, D, E and F form a 5-cycle, and A is adjacent to B, C, D, E and F.

Hence, we need 3 colors to color B, C, D, E and F, and one more color to color A, so we need at least 4 colors.

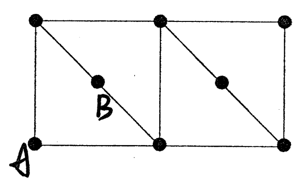
So the chromatic number is 4.

7. Find the length of the shortest path from A to Z in the following weighted graph. Shade in all shortest paths in the figure (there are two copies for your convenience).



The number for vertices are the number in the label of vertices.

Is the following graph Hamiltonian? Explain your answer.



No.

Suppose it is Hamiltonian, so it has a Hamiltonian cycle C.

Since vertex A and B are of degree 2, the 2 edges incident with A and B are in C, but it implies that C contains a 4-cycle as a subgraph.

So C only passes through 4 vertices, contradicting the definition of Hamiltonian cycles.

So the graph is not Hamiltonian.

8. Is the following matrix diagonalizable? Explain your answer.

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Find eigenvalues:

$$\begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3-\lambda)^3 = 0$$

$$\Rightarrow \quad \lambda = 3.$$

If $\lambda = 3$ has 3 eigenvectors then the matrix is diagonalisable.

To see how many eigenvectors we have:

Solve

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

But this has only one free variable, hence $\lambda = 3$ has only one eigenvector.

So the matrix is <u>not</u> diagonalisable.

9. Diagonalize the matrix

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}.$$

let $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$

Eigenvalues: $0 = |A - \lambda I_2| = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$

So $\lambda = 1, 2$

Eigenvectors: $\lambda = 1$: $\left(\begin{array}{cc|c} 2 & -2 & 0 \\ 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ so: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 2$: $\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ So: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

let $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $P^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

then $A = PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

Use your answer to the first part to find a formula for $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\begin{aligned}
A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= (PDP^{-1})^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = PD^n P^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= PD^n \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = PD^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= P\begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = P\begin{pmatrix} -1 \\ 2^n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2^n \end{pmatrix} \\
&= \begin{pmatrix} -1 + 2^{n+1} \\ -1 + 2^n \end{pmatrix}
\end{aligned}$$

What second order linear recurrence relation did you just solve?

$a_{n+2} = 3a_{n+1} - 2a_n$, $a_0 = 0, a_1 = 1$

10. Solve the following linear programming problem via the simplex method. Show all work.

$$\begin{array}{lll} \text{Maximize} & z = 8x + 10y & \longrightarrow z - 8x - 10y = 0 \\ \text{subject to} & 2x + y \leq 50 & \longrightarrow 2x + y + S_1 = 50 \\ & x + 2y \leq 70 & \longrightarrow x + 2y + S_2 = 70 \\ & x, y \geq 0 & \longrightarrow x, y, S_1, S_2 \geq 0 \end{array}$$

	z	x	y	S_1	S_2	RHS
	1	-8	-10	0	0	0
	0	2	1	1	0	50
	0	1	$\boxed{2}$	0	1	70
① + 5③	1	-3	0	0	5	350
② - ③/2	0	$\boxed{3/2}$	0	1	-1/2	15
③/2	0	1/2	1	0	1/2	35
① + 2②	1	0	0	2	4	380
$\frac{2}{3}$②	0	1	0	2/3	-1/3	10
③ - $\frac{1}{3}$②	0	0	1	-1/3	1/6	30

Pivot about boxed number.

Done as all of first row is positive.

So maximum is $z = 380$

when $x = 10$, $y = 30$.