Scores: 1 2 3 4 5 6 7 8 9 10 E Name Prof. M

Section L___

Mathematics 2602

Final Exam

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13 December 2011

1. State the *principle of mathematical induction.*

Say we have a mathematical statement that depends on an integer n and that

① The statement is true for $n = n_0$

② Whenever the statement is true for $n = k$, it is true for $n = k+1$.

Then the statement is true for all $n \geq n_0$.

State the definitions of *eigenvalue* and *eigenvector.*

Say A is a matrix, v is a nonzero vector, and $\lambda \in \mathbb{R}$. If

$$Av = \lambda v$$

then v is an eigenvector for A and λ is the associated eigenvalue.

2. Recall that the Fibonacci numbers are defined by the recursion relation

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1.$$

Use the principle of mathematical induction to show that

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n$$

for all $n \geq 1$.

Base case: $n=1$

$$F_0 F_2 = 0 \cdot 1 = 0$$

$$F_n^2 + (-1)^n = 1^2 - 1 = 0 \quad \checkmark$$

Assume true for $n=k$:

$$F_{k-1}F_{k+1} = F_k^2 + (-1)^k.$$

Show true for $n=k+1$:

$$\begin{aligned}
F_k F_{k+2} &= F_k(F_k + F_{k+1}) \\
&= F_k^2 + F_k F_{k+1} \\
&= F_{k-1}F_{k+1} - (-1)^k + F_k F_{k+1} \\
&= F_{k+1}(F_k + F_{k+1}) + (-1)^{k+1} \\
&= F_{k+1}F_{k+2} + (-1)^{k+1}
\end{aligned}$$

By induction, the statement is proven.

3. Solve the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} + 3^n$$

with initial conditions $a_0 = 2$ and $a_1 = 10$.

First, we solve completely

$$a_n^{(h)} = a_{n-1}^{(h)} + 2a_{n-2}^{(h)}$$

Char. poly.: $x^2 - x - 2 = 0$

$$(x-2)(x+1)$$

$$x = 2, -1$$

$$a_n^{(h)} = k_1 2^n + k_2(-1)^n.$$

Next, we find one solution to a_n.

Guess: $c3^n$.

Check: $c3^n = c3^{n-1} + 2c3^{n-2} + 3^n$

$$9c = 3c + 2c + 9$$

$$4c = 9$$

$$c = 9/4$$

$$\rightsquigarrow a_n = 9/4\, 3^n = \frac{3^{n+2}}{4}.$$

Adding, the general solution to a_n is:

$$a_n = k_1 2^n + k_2(-1)^n + \frac{3^{n+2}}{4}$$

Using $a_0 = 2$, $a_1 = 10$ we solve:

$$k_1 = 1,\ k_2 = -5/4.$$

Final answer:

$$a_n = 2^n - 5/4(-1)^n + \frac{3^{n+2}}{4}.$$

4. Show that $\log(n!) = \mathcal{O}(n \log n)$.

$$\log(n!) = \log(n(n-1)\cdots 1)$$
$$\leq \log n^n$$
$$= n \log n$$

Taking $c = 1$, $n_0 = 1$ we have

$$\log(n!) \leq n \log n \quad n \geq n_0.$$

So $\log(n!) = \mathcal{O}(n \log n)$.

What is the smallest number of people you would need to gather if you wanted to ensure that at least three people have the same first and last initials (for example, three people with the initials DM, or three people with the initials AB, etc.)? Explain your answer.

We need $2 \cdot 26^2 + 1$ people.

By the pigeonhole principle there must

be at least $\left\lceil \frac{2 \cdot 26^2 + 1}{26^2} \right\rceil = 3$

people with the same initials.

5. A candy store has bubble gum balls in 7 colors. How many ways are there to buy 15 gum balls if you buy at least one of each color?

15 marbles, 7 boxes, with repetition.

No choice on first 7 marbles, so really

8 marbles, 7 boxes

$$\binom{14}{6}$$

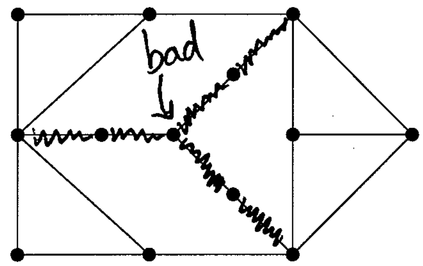
There are four committees, each with five people. For every pair of distinct committees, there is one person that is on both committees. Exactly one person is on three committees and nobody is on four committees. How many people are on at least one committee?

By the principle of inclusion-exclusion:

$$5 \cdot 4 - \binom{4}{2} \cdot 1 + 1 - 0$$

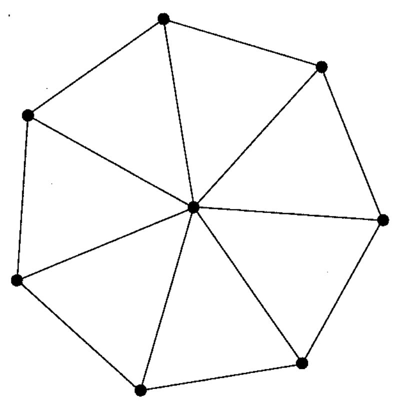
$$= 20 - 6 + 1 = 15.$$

6. Decide whether or not the following graph is Hamiltonian. If it is, exhibit a Hamiltonian cycle. If it is not, explain why not.



It is not. Any Hamiltonian cycle must have both edges incident to a degree 2 vertex, but cannot have 3 edges incident to the same vertex. As shown, this is a contradiction.

What is the chromatic number of the following graph? Explain your answer.



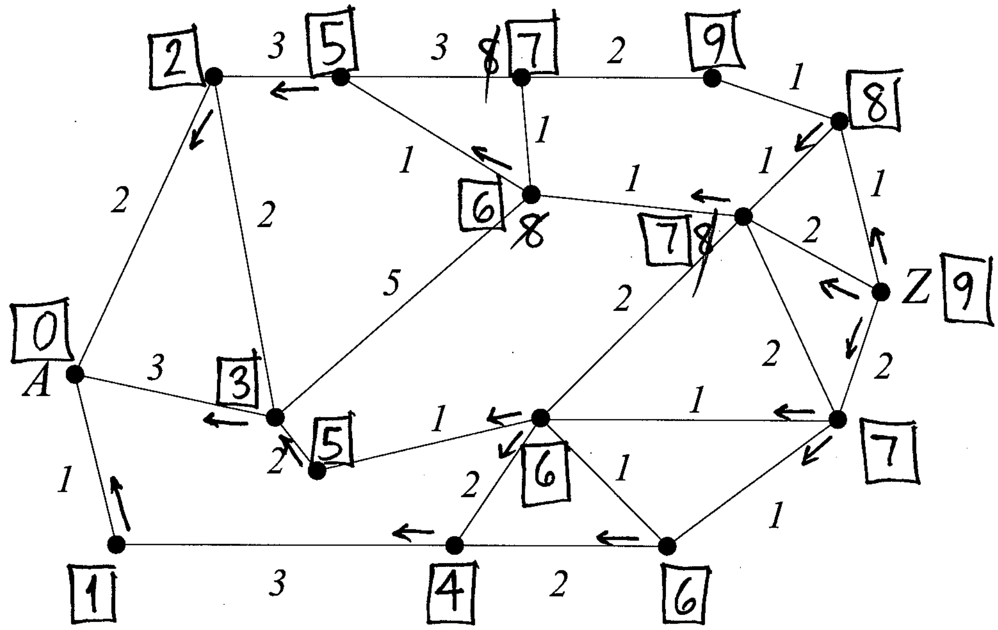
$\chi \le 4$ by planarity

$\chi \geq 4$: The outer 7-gon needs 3 colors (true for any odd cycle).

Then we need a 4^{th} color for center.

So $\boxed{\chi = 4}$

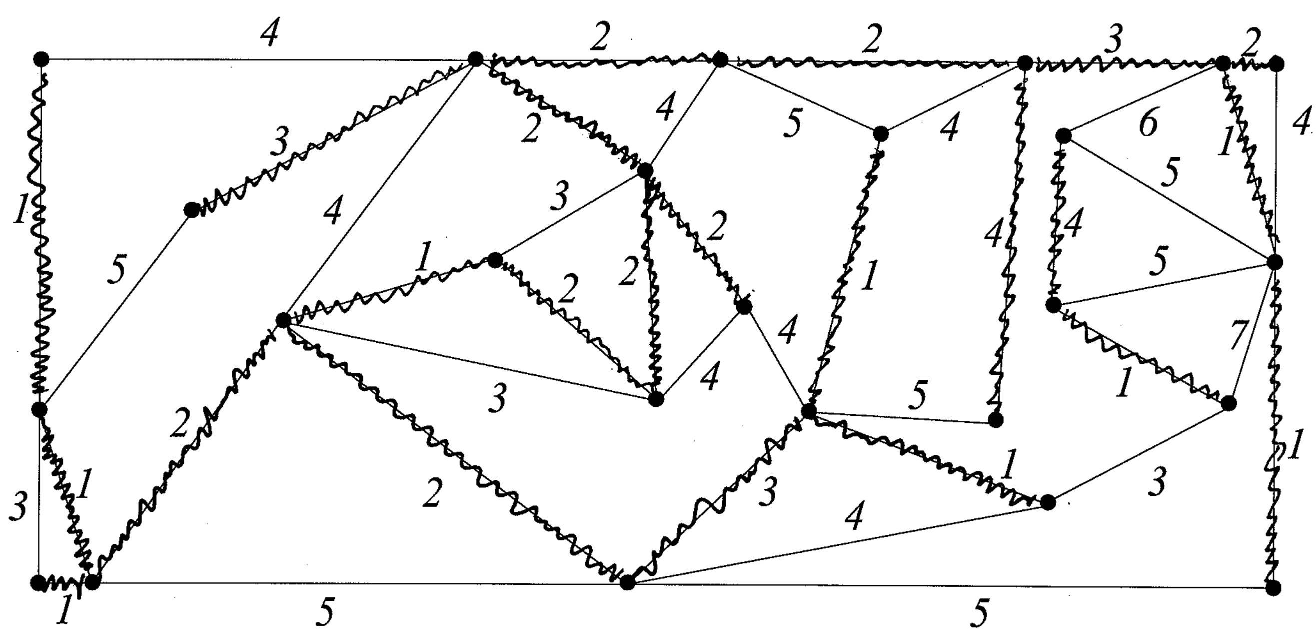
7. Find the length of the shortest path from A to Z in the following weighted graph.



How many paths from A to Z are there that have the shortest length?

5

Find a minimal spanning tree for the following weighted graph. Use Prym's algorithm, starting from the bottom left corner. Shade in your tree in the diagram.



8. Is the following matrix diagonalizable? Explain your answer.

$$\begin{pmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues: 2, 1, 2 (matrix is upper Δ).

Need to check if 2 has two independent eigenvectors:

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution: $y=0$, x and z free.
(= x-z plane).

Since the solution space is two-dimensional, there are independent eigenvectors, for example $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

So the matrix is diagonalizable.

9. Diagonalize the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$$

Eigenvalues: $(4-\lambda)(3-\lambda) - 6$

$\lambda^2 - 7\lambda + 6$

$(\lambda-6)(\lambda-1)$

$\lambda = 6, 1$

Eigenvectors: $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \end{pmatrix}$

$4x + 2y = x$ $\quad$ $4x + 2y = 6x$

$3x = -2y$ $\quad$ $x = y$

$\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

Find A^{100}. Your answer should be a 2×2 matrix, but you do not need to simplify the entries.

$$A^{100} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 6^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3/\cancel{6}5 & 2/\cancel{6}5 \\ -1/\cancel{6}5 & 1/\cancel{6}5 \end{pmatrix}$$

$$= \begin{pmatrix} 6^{100} & -2 \\ 6^{100} & 3 \end{pmatrix}\begin{pmatrix} 3/5 & 2/5 \\ -1/5 & 1/5 \end{pmatrix}$$

$$= \begin{pmatrix} 6^{100} \cdot 3/5 + 2/5 & 6^{100} \cdot 2/5 - 2/5 \\ 6^{100} \cdot 3/5 - 3/5 & 6^{100} \cdot 2/5 + 3/5 \end{pmatrix}$$

10. Solve the following linear programming problem via the simplex method. Show all work.

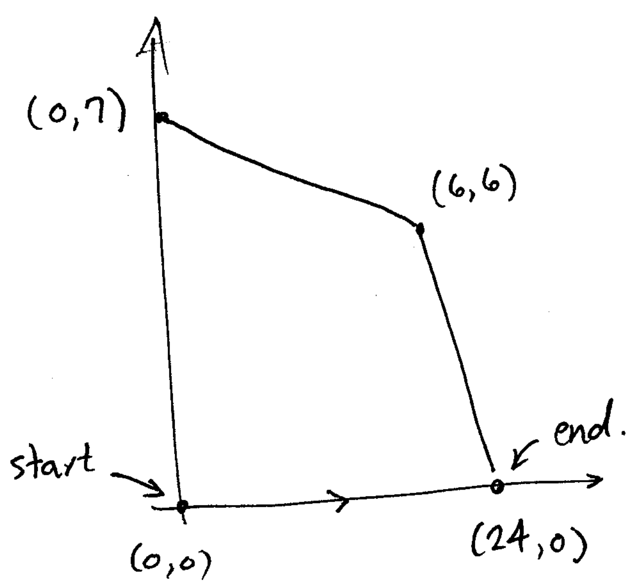
$$\begin{array}{rl} \text{Maximize} & z = x + y \\ \text{subject to} & x + 3y \leq 24 \\ & x + 6y \leq 42 \\ & x, y \geq 0 \end{array}$$

z	x	y	s_1	s_2	RHS
1	-1	-1	0	0	0
0	1	3	1	0	24
0	1	6	0	1	42

z	x	y	s_1	s_2	RHS
1	0	2	1	0	24
0	1	3	1	0	24
0	0	3	-1	1	18

$z = 24$ at $x = 24$, $y = 0$.

Draw the feasible region, and show how the basic solution changes during the simplex method.



Extra credit (5 points). Show that

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix},$$

where F_n is the nth Fibonacci number, as in Problem 2. Use this to find a non-recursive formula for F_n. What is the relationship between the characteristic polynomial of a second order linear homogeneous recurrence relation and the characteristic polynomial of a 2×2 matrix?

Idea: Diagonalize A: $A = CDC^{-1}$

Find formula for A^n: CD^nC^{-1}.

Similarly, for $a_n = ra_{n-1} + sa_{n-2}$

take $A = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}$

The char. poly. of A is

$(r-\lambda)(-\lambda) - s$

$\lambda^2 - r\lambda - s$.