Scores: 1 2 3 4 5 6 7 8 9 10 Name Prof. M

Section K___

Mathematics 2602

Midterm 1

Prof. Margalit

31 January 2013

1. Write Y if the statement is a proposition and N if it is not a proposition.

$x + 2 = 11$

N

Find an x so that $x + 3 = 5$.

N

Write T if the proposition is true and F if the proposition is false.

If $1 + 1 = 2$ or $1 + 1 = 3$ then $1 + 1 = 4$ or $1 + 1 = 5$.

F

$((1 + 1 = 2) \vee (1 + 1 = 3)) \wedge \neg(1 + 1 = 4)$

T

2. Complete the following truth table.

p	q	$\neg p$	$p \rightarrow q$	$q \vee \neg p$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

What can you conclude from the truth table?

$$(p \rightarrow q) \equiv (q \vee \neg p)$$

3. Show that $p \rightarrow (q \rightarrow r)$ is equivalent to $(p \wedge q) \rightarrow r$ without using truth tables.
Hint: Use the problem on the previous page.

$$p \rightarrow (q \rightarrow r)$$

$$\equiv \neg p \vee (q \rightarrow r)$$

$$\equiv \neg p \vee (\neg q \vee r)$$

$$\equiv (\neg p \vee \neg q) \vee r$$

$$\equiv \neg (p \wedge q) \vee r$$

$$\equiv (p \wedge q) \rightarrow r$$

4. Let $L(x, y)$ be the statement "x loves y." Match the following English sentences with the formal propositions below.

Everybody loves somebody.

F

There is somebody whom everybody loves.

G

Nobody loves everybody.

J

A.	$\forall x L(x, x)$	*F.*	$\forall x \exists y L(x, y)$
B.	$\exists x L(x, y)$	*G.*	$\exists x \forall y L(y, x)$
C.	$\neg(\exists x L(x, Me))$	*H.*	$\exists x \forall y L(x, y)$
D.	$\forall y \exists x L(x, y)$	*I.*	$\exists x \neg L(x, Me)$
E.	$\exists x \forall y L(x, y)$	*J.*	$\forall x \exists y \neg L(x, y)$

5. Determine the truth values of the following propositions. Write T if the proposition is true and F if the proposition is false.

$\forall x \exists y (y = x^2) \quad x, y \in \mathbb{R}$

T

$\forall x (x \neq 0 \rightarrow \exists y (xy = 1)) \quad x, y \in \mathbb{R}$

T

$\exists n \forall m (mn = m) \quad m, n \in \mathbb{Z}$

T

6. Recall the following rules of inference.

Rule #1	$(p \wedge (p \to q)) \to q$	Rule #6	$(p \wedge q) \to p$
Rule #2	$(\neg q \wedge (p \to q)) \to \neg p$	Rule #7	$((p) \wedge (q)) \to (p \wedge q)$
Rule #3	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Rule #8	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$
Rule #4	$((p \vee q) \wedge \neg p) \to q$	Rule #9	$\forall x P(x) \to P(c)$
Rule #5	$p \to (p \vee q)$	Rule #10	$P(c) \to \exists x P(x)$

Which rules of inference are used in the following argument? Explain your answer by explicitly stating which rules you are applying to which propositions.

(1) Everyone is either dreaming or hallucinating. (2) If I am hallucinating, then I see unicorns and ponies. (3) I am not dreaming. (4) Therefore I see ponies.

(5) I am dreaming or hallucinating — Rule #9 applied to (1)

(6) I am hallucinating — Rule #4 applied to (5),(3)

(7) I see unicorns and ponies — Rule #1 appl. to (2),(6)

(4) I see ponies — Rule #6 applied to (7)

7. Prove that if x and y are real numbers with xy irrational, then at least one of x and y is irrational.

We will prove the contrapositive: if x, y rational, then xy is rational.

x, y rational

$\rightarrow \quad x = \frac{p}{q}, \quad y = \frac{r}{s} \qquad p, q, r, s \in \mathbb{Z}$

$\rightarrow \quad xy = \frac{pr}{qs}$

Since $pr, qs \in \mathbb{Z}$, xy is rational.

8. Prove that if integers m and n have opposite parity (that is, one is even and one is odd), then $mn + m + n$ is odd.

Without loss of generality, m is even and n is odd.

$\longrightarrow \; m = 2p, \; n = 2q+1 \qquad p, q \in \mathbb{Z}.$

$$\longrightarrow \; mn + m + n = 2p(2q+1) + 2p + 2q + 1$$

$$= 2\big(p(2q+1) + p + q\big) + 1$$

which is odd (since $p(2q+1+p+q) \in \mathbb{Z}$).

9. A function $f : \mathbb{R} \to \mathbb{R}$ is *rational* if it is the quotient of two polynomials. Prove or disprove that $f(x) = \sqrt[4]{x}$ is a rational function.

We proceed by contradiction.

Assume $\sqrt[4]{x}$ is rational.

$\rightarrow \sqrt[4]{x} = \dfrac{p(x)}{q(x)}$ $\quad$ $p(x), q(x)$ are polynomials.

$\rightarrow x = \dfrac{(p(x))^4}{(q(x))^4}$

$\rightarrow x(q(x))^4 = (p(x))^4$

The left hand side is a polynomial of odd degree and the right hand side is a polynomial of even degree.

This is a contradiction.

10. Prove or disprove that every positive integer is equal to the sum of the squares of two integers.

We disprove by counterexample.

We claim that 3 is not the sum of two squares. Indeed, the only squares ≤ 3 are 0 and 1.

But none of $0+0$, $0+1$, $1+1$, $1+0$ is equal to 3.