Scores: 1 2 3 4 5 6 7 8 9 10   Name PROF. M

Section K___

Mathematics 2602

Midterm 2

Prof. Margalit

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1. Choose the correct definition of "f is $\mathcal{O}(g)$."

   A. $\forall c \forall n_0 (n \geq n_0 \rightarrow f(n) > cg(n))$

   B. $\forall c \forall n_0 (n \geq n_0 \rightarrow f(n) \leq cg(n))$

   C. $\exists c \exists n_0 (n \geq n_0 \rightarrow f(n) \geq cg(n))$

   D. $\exists c \exists n_0 (n \geq n_0 \rightarrow f(n) \leq cg(n))$

D

Choose the correct statement of the principle of mathematical induction, or choose "None of the above." In each statement $P(n)$ is a propositional function.

A. If $P(k) \rightarrow P(k+1)$ for $k \geq n_0$, then $P(n)$ is true for all $n \geq n_0$.

B. If $P(n_0)$, $P(k)$, and $P(k+1)$ are true, then $P(n)$ is true for all $n \geq n_0$.

C. If $P(n_0)$ is true and $P(k) \rightarrow P(k+1)$ for $k \geq n_0$, then $P(n)$ is true for $n \geq n_0$.

D. None of the above.

C

2. Consider the sequence given by

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_n &= 2a_{n-1} + a_{n-2}, \quad n \geq 2 \end{aligned}$$

What is a_5?

0 1 2 5 12 29

$a_5 = 29$

Give a recursive definition of the sequence $a_n = (n!)^2$.

$a_0 = 1$

$a_n = n^2 a_{n-1} \quad n \geq 1.$

3. Give a big-O estimate for the number of additions and multiplications needed in order to multiply two $n \times n$ matrices. In pseudocode, the usual algorithm is:

procedure *matrix multiplication*$((a_{i,j}), (b_{i,j}) : n \times n$ real-valued matrices$)$
for $i := 1$ to n
    for $j := 1$ to n
        $c_{i,j} := 0$
        for $k := 1$ to n
            $c_{i,j} := c_{i,j} + a_{i,k}b_{k,j}$
return $(c_{i,j})$

A. $\mathcal{O}(3n^2)$

B. $\mathcal{O}(n^3)$

C. $\mathcal{O}(\log n^3)$

D. $\mathcal{O}(3^n)$

B

What does it mean for a function $f(n)$ to be $\mathcal{O}(1)$?

A. $f(n)$ is bounded

B. $f(n) = 1$ for large enough n

C. $\lim_{n\to\infty} f(n) = 0$

D. $\lim_{n\to\infty} f(n) = 1$

A

4. Find the smallest integer n so that the following function is $\mathcal{O}(x^n)$.

$$\frac{x^4 + 5\log x}{x^4 + x^2 + x + 1}$$

$n = 0$

Arrange the following functions in a list so that each function is big-O of the next.

$$(1.5)^n, n^{100}, (\log n)^3, \sqrt{n}\log n, 10^n, (n!)^2, n^{99} + n^{98}$$

$(\log n)^3 < \sqrt{n} \log n < n^{99} + n^{98}$

$< n^{100} < 1.5^n < 10^n$

$< (n!)^2$

5. Recall the bubble sort algorithm:

procedure *bubble sort*$(a_1, \ldots, a_n$: real numbers with $n \geq 2)$
for $i := 1$ to $n - 1$
  for $j := 1$ to $n - i$
    if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}
return $(a_1, \ldots, a_n)$

Apply this algorithm to the sequence $(12, 1, 17, 3, 11)$. Use a new column for each interchange. You do not need to use all of the columns.

12	1	1	1	1	1		
1	12	12	12	3	3		
17	17	3	3	12	11		
3	3	17	11	11	12		
11	11	11	17	17	17		

What is the largest number of swaps needed to perform the bubble sort algorithm on a list of 5 numbers?

$4 + 3 + 2 + 1 = 10$

6. Show that $9x + 10$ is $\mathcal{O}(x^2)$ by finding a specific c and n_0 as in the definition of "f is $\mathcal{O}(g)$."

$C = 19, \quad n_0 = 1$

$9n + 10 \leq 9n^2 + 10n^2 = 19n^2 \quad n \geq 1.$

or

$C = 1, \quad n_0 = 10$

$x \geq 10 \longrightarrow (x-10)(x+1) \geq 0$

$\longrightarrow x^2 - 9x - 10 \geq 0$

$\longrightarrow x^2 \geq 9x + 10$

Which of the following describes the relationship between $9x + 10$ and x^2?

A. $9x + 10 \prec x^2$

B. $9x + 10 \succ x^2$

C. $9x + 10 \asymp x^2$

D. None of the above.

A

7. Determine

$$\lim_{n\to\infty} \frac{3^n}{n^n}.$$

Justify your answer.

The limit is 0 because

$$\lim_{n\to\infty} \frac{3^n}{n^n} = \lim_{n\to\infty} \left(\frac{3}{n}\right)^n$$

$$\leq \lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0.$$

Which of the following describes the relationship between 3^n and n^n?

A. $3^n \prec n^n$

B. $3^n \succ n^n$

C. $3^n \asymp n^n$

D. None of the above.

A.

8. Use induction to prove that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

for $n \geq 1$.

Base step: $1 \cdot 1! = (1+1)! - 1$

$1 = 2 - 1$ ✓

Assume $1 \cdot 1! + \cdots + k \cdot k! = (k+1)! - 1$

Then $1 \cdot 1! + \cdots + k \cdot k! + (k+1)(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)!\,(1 + (k+1)) - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1$

9. Consider the sequence defined by

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 3 \\
a_n &= a_{n-1}^3 + a_{n-2} + 1, \quad n \geq 2.
\end{aligned}
$$

Use induction to show that a_n is odd for all $n \geq 0$. You may use basic facts about even and odd numbers.

Base cases: $a_0 = 1$, $a_1 = 3$ both odd.

Assume $a_0, \ldots, a_{n-1}$ odd.

Then

$$
\begin{aligned}
a_n &= a_{n-1}^3 + a_{n-2} + 1 \\
&= (\text{odd})^3 + \text{odd} + 1 \\
&= \text{odd} + \text{odd} + 1 \\
&= \text{even} + 1 \\
&= \text{odd}.
\end{aligned}
$$

10. Consider the following recursively-defined set S.

Basis step: $\binom{1}{0} \in S$

Recursive step: If $(m,n) \in S$ then

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} \in S \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} \in S$$

List the elements of S obtained in the first application of the recursive step.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Use structural induction to show that if $(m,n) \in S$ then m is odd and n is even.

Base step: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}$ ✓

Recursive step: Suppose $m = 2a+1$, $n = 2b$ $a, b \in \mathbb{Z}$.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2a+1 \\ 2b \end{pmatrix} = \begin{pmatrix} 2a+4b+1 \\ 2b \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2a+1 \\ 2b \end{pmatrix} = \begin{pmatrix} 2a+1 \\ 4a+2b+2 \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}$$