Scores: 1 2 3 4 5 6 7 8 9 10 Name Prof. M

Section K___

Mathematics 2602

Midterm 3

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1. How many different three-letter initials can people have? Do not simplify your answer.

$$26^3$$

Assume nobody has more than 150,000 hairs on his or her head and that the population of Atlanta is 432,427. What is the largest number N so that you can conclude there are N people in Atlanta with the same number of hairs on their heads?

$$\left\lceil \frac{432,427}{150,000} \right\rceil = 3$$

2. How many three-digit numbers (100-999) contain a 7? Your answer should be a number.

$$900 - 8 \cdot 9 \cdot 9 = 252$$

How many numbers between 1 and 1000 are divisible by 3 and/or 101? Your answer should be a number.

$$\left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{101} \right\rfloor - \left\lfloor \frac{1000}{303} \right\rfloor$$

$$= 333 + 9 - 3 = 339$$

3. How many different strings can be made from the letters in ABRACADABRA using all of the letters? Do not simplify your answer.

$$11!/5!2!2!$$

A computer network consists of 100 computers. Each computer is directly connected to at least one other computer. Can we conclude that there are two computers directly connected to the same number of other computers? Explain your answer.

Yes. There are 100 computers, each with 1 to 99 connections. By the pigeonhole principle, two have the same number of connections.

4. In how many ways can a photographer arrange the seven dwarfs in a line if Dopey and Grumpy insist on standing next to each other? Do not simplify your answer.

$2 \cdot 6!$

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint*: First position the men and then consider the possible positions for the women.] Do not simplify your answer.

$8! \binom{9}{5} 5!$

5. Find the expansion of $(x+y)^4$. Simplify your answer.

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Find the coefficient of x^3 in $\left(x^3 - \dfrac{3}{x}\right)^5$. Simplify your answer.

K^{th} term: $\binom{5}{K}(x^3)^K\left(-\frac{3}{x}\right)^{5-K}$

$$(-3)^{5-K}\binom{5}{K}x^{4K-5}$$

x^3 is $K=2$ term.

$$(-3)^3\binom{5}{3} = -270$$

6. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop? Do not simplify your answer.

$$\binom{32}{12}$$

How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 19$$

if each x_i is a nonnegative integer and $x_1 \geq 4$? Do not simplify your answer.

Same as: $X_1' + X_2 + X_3 = 15$ $\quad X_1' = X_1 - 4.$

$$\binom{17}{2}$$

7. Find a recursive formula for the number of n-letter strings in the English alphabet with an even number of vowels (there are 21 consonants and 5 vowels).

$$a_1 = 21$$

$$a_n = 21 \cdot a_{n-1} + 5 \cdot (26^{n-1} - a_{n-1})$$

Which of the following are linear recurrence relations? Select all that apply.

$A.$ $a_n = a_{n-1} + a_{n-2} + a_{n-3}$

$B.$ $a_n = a_{n-1}^2$

$C.$ $a_n = a_{n-1} + a_{n+1}$

$D.$ $a_n = a_{n-1} + a_{n-2}a_{n-3}$

8. Solve the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$.

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$r = 3, 3$$

$$\rightsquigarrow a_n = c3^n + dn3^n$$

$$a_0 = 1 = c$$

$$a_1 = 6 = 3 + 3d$$

$$\rightsquigarrow a_n = 3^n + n3^n$$

9. Solve the recurrence relation

$$a_n = 3a_{n-1} - 2$$

with initial condition $a_0 = 3$.

① General solution to $a_n = 3a_{n-1}$

$$q_n = c3^n$$

② Particular solution to $a_n = 3a_{n-1} - 2$

$$p_n = b$$

$$b = 3b - 2$$

$$b = 1$$

③ Add: $a_n = c3^n + 1$

④ Solve $a_0 = 3 = c + 1$

$$a_n = 2 \cdot 3^n + 1$$

10. Give one derangement of the set $\{1, 2, 3, 4\}$.

$$\{4, 3, 2, 1\}$$

Each lottery ticket has all of the digits 0 through 9 written in some order. A ticket wins if any of the following holds: *(i)* it begins with 987, *(ii)* it contains 45 in the fifth and sixth spots (in that order), or *(iii)* it ends in 123. How many different winning tickets are there? Do not simplify your answer.

$A_1 = \{\text{tix starting } 987\}$

$A_2 = \{\text{tix with } 45 \text{ in middle}\}$

$A_3 = \{\text{tix ending } 123\}$

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ &\quad + |A_1 \cap A_2 \cap A_3| \\ &= 7! + 8! + 7! - 5! - 4! - 5! + 2! \end{aligned}$$