Scores: 1 2 3 4 5 6 7 8 9 10 Name Prof. M

Section K___

Mathematics 2602

Final Exam

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2 May 2013

1. Determine the truth value of the following proposition:

$$(\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x = y^2)) \rightarrow (1 + 1 = 3)$$

A. True

B. False

C. Inconclusive

D. The statement is not a proposition.

Give a big-O estimate for the complexity (number of swaps) of the bubble sort algorithm:

procedure *bubble sort*$(a_1, \ldots, a_n$: real numbers with $n \geq 2)$
for $i := 1$ to $n - 1$
  for $j := 1$ to $n - i$
    if $a_j > a_{j+1}$ **then** swap a_j and a_{j+1}
return $(a_1, \ldots, a_n)$

A. $\mathcal{O}(n)$

B. $\mathcal{O}(n^2)$

C. $\mathcal{O}(\log n)$

D. $\mathcal{O}(n \log n)$

Find a recursive formula for the number of n-letter strings in the English alphabet with an even number of vowels (there are 21 consonants and 5 vowels).

A. $a_n = 16a_{n-1} + 5 \cdot 26^{n-1}$

B. $a_n = 21a_{n-1} + 5 \cdot 26^{n-1}$

C. $a_n = 21a_{n-1} + 5a_{n-2}$

D. $a_n = 21a_{n-1} + 5 \cdot (26^{n-1} - a_{n-2})$

2. Recall the following rules of inference.

Rule #1 $(p \wedge (p \rightarrow q)) \rightarrow q$	Rule #6 $(p \wedge q) \rightarrow p$
Rule #2 $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Rule #7 $((p) \wedge (q)) \rightarrow (p \wedge q)$
Rule #3 $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Rule #8 $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$
Rule #4 $((p \vee q) \wedge \neg p) \rightarrow q$	Rule #9 $\forall x P(x) \rightarrow P(c)$
Rule #5 $p \rightarrow (p \vee q)$	Rule #10 $P(c) \rightarrow \exists x P(x)$

Consider the following statements.

(1) Everyone who bowls eats cookies. (2) Jan bowls.
Conclusion: There is someone who bowls and eats cookies.

Rewrite all three statements using the notation of propositional logic and the functions:

$$B(x) = x \text{ bowls.}$$
$$C(x) = x \text{ eats cookies.}$$

(1) $\forall x \, (B(x) \rightarrow C(x))$

(2) $B(\text{Jan})$

Conclusion: $\exists x \, (B(x) \wedge C(x))$

Derive the conclusion from the two given statments and the rules of inference. (You do not need to use all of the lines.)

	Statement	Rule applied	Statements used
(3)	$B(\text{Jan}) \rightarrow C(\text{Jan})$	9	1
(4)	$C(\text{Jan})$	1	2,3
(5)	$B(\text{Jan}) \wedge C(\text{Jan})$	7	2,4
(6)	$\exists x \, (B(x) \wedge C(x))$	10	5
(7)			

3. Prove that $16x^2 - 2x + 5$ is $\mathcal{O}(x^3)$ by finding a c and n_0 as in the definition of big-O. Make sure to justify your choice of c and n_0.

$$c = 21, \quad n_0 = 1.$$

$$16x^2 - 2x + 5 \leq 16x^2 + 5 \leq 16x^3 + 5x^3 = 21x^3.$$

$$x \geq 1.$$

Arrange the following functions in a list so that each function is big-O of the next:

$$n^{3/2}\ ,\ 99999\log n\ ,\ n\log n\ ,\ (n!)^2\ ,\ 2^n\ ,\ n^2\ ,\ 3^{n-1}$$

$$99999 \log n < n \log n < n^{3/2} < n^2$$

$$< 2^n < 3^{n-1} < (n!)^2$$

4. Use induction to prove that for every positive integer n we have:

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Base case: $n = 1$

$$1 \cdot 2 = \frac{1(2)(3)}{3} \quad \checkmark$$

Assume: $1 \cdot 2 + \cdots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$

Then: $1 \cdot 2 + \cdots + (k+1)(k+2)$

$$= 1 \cdot 2 + \cdots + k(k+1) + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$$

$$= \frac{(k+3)(k+1)(k+2)}{3}$$

$$= \frac{(k+1)(k+2)(k+3)}{3}$$

5. Solve the recurrence relation given by $a_0 = 4$, $a_1 = 10$ and

$$a_n = 6a_{n-1} - 9a_{n-2} + 4n, \quad n \geq 2.$$

① Solve $q_n = 6q_{n-1} - 9q_{n-2}$

$$r^2 - 6r + 9 = 0$$

$$\rightsquigarrow r = 3, 3$$

$$q_n = a3^n + bn3^n$$

② Find particular soln $p_n = mn + b$:

$$mn + b = 6(m(n-1) + b) - 9(m(n-2) + b) + 4n$$

$$\rightsquigarrow m = 1, b = 3$$

③ Add: $a_n = a \cdot 3^n + b \cdot n \cdot 3^n + (n+3)$

④ Use a_0, a_1 to solve for a, b :

$$a_n = 3^n + n \cdot 3^n + (n+3)$$

6. A bagel shop has 8 kinds of bagels. How many ways are there to choose two dozen bagels with at least one of each kind if there are only two sesame bagels (and many of the other kinds)? Do not simplify your answer.

Need to choose 24-8 bagels

One sesame: 6 bars, 16 stars $\rightsquigarrow \binom{22}{6}$

Two sesame: 6 bars, 15 stars $\rightsquigarrow \binom{21}{6}$

$$\binom{21}{6} + \binom{22}{6}$$

How many integers from 1 to 100 are either odd, a square, divisible by 7, or some combination of these? Do not simplify.

$A = \{\text{odds}\}$, $B = \{\text{squares}\}$

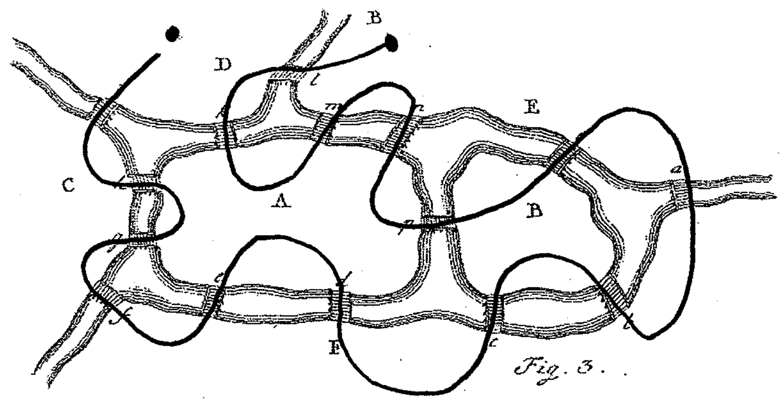
$C = \{\text{mult. of } 7\}$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= 50 + 10 + 14 - 5 - 1 - 7 + 1$$

$$= 62$$

7. Is it possible to take a walk that crosses each bridges exactly once, if one is not required to return to the starting point? Explain your answer.



Yes. See picture.

What if one is required to return to the starting point? Explain your answer.

No. The corresponding graph has a vertex of odd degree corresponding to land mass D.

Which of the following graphs are Hamiltonian? Select all that apply.

A. K_2

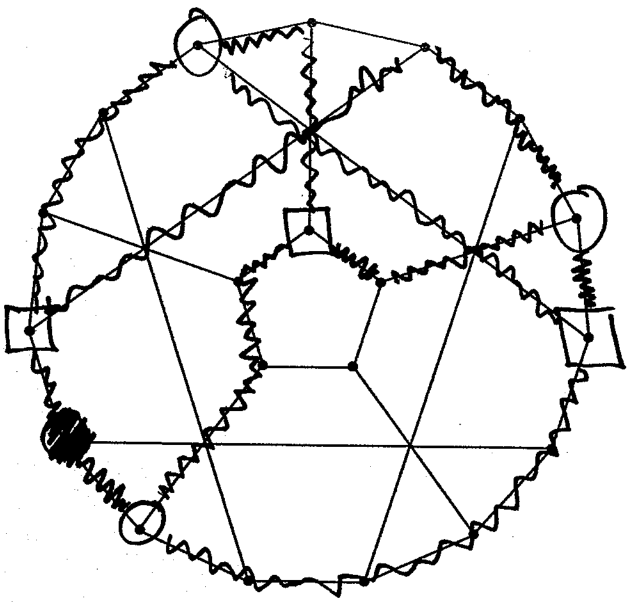
B. K_{100}

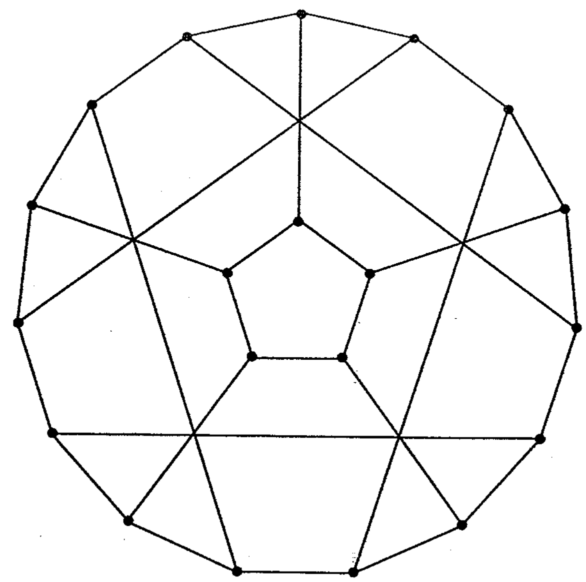
C. W_{100}

D. $K_{3,3}$

E. $K_{100,101}$

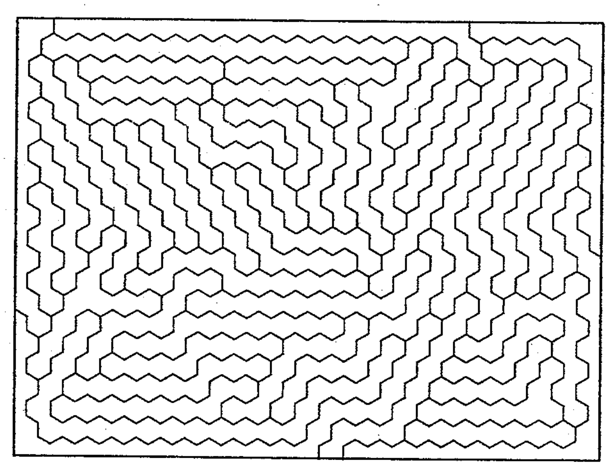
8. Is the following graph planar? Justify your answer. (Two copies provided for convenience.)





No. A subdivision of $K_{3,3}$ is shaded.

What is the chromatic number of the following map? Justify your answer.

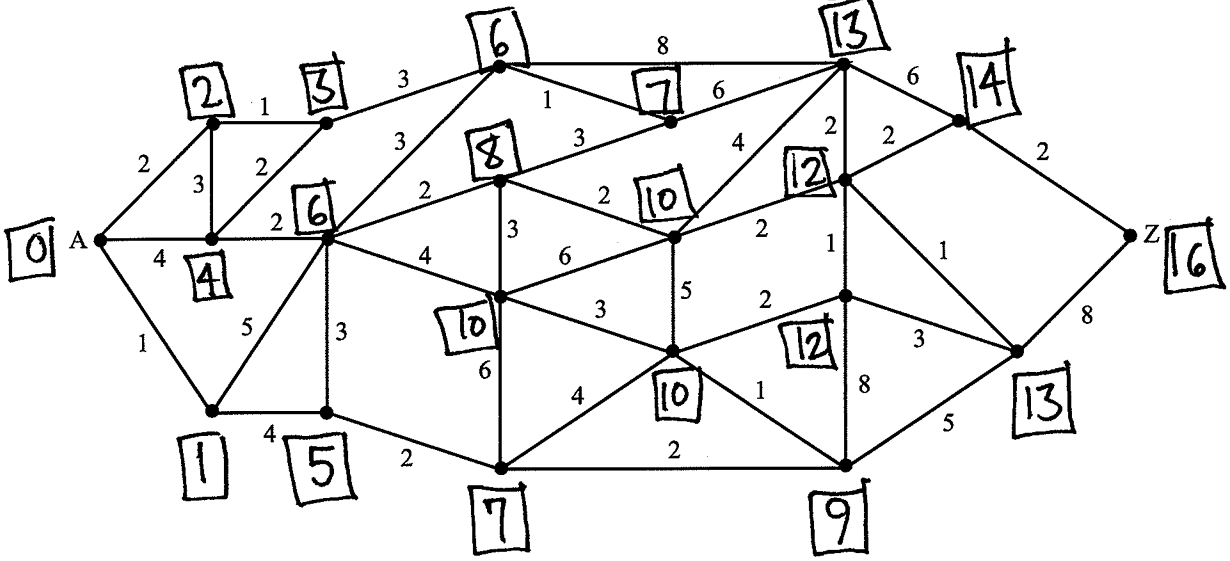


$\chi \leq 4$ by planarity

$\chi \geq 4$ since there are 4 mutually adjacent regions

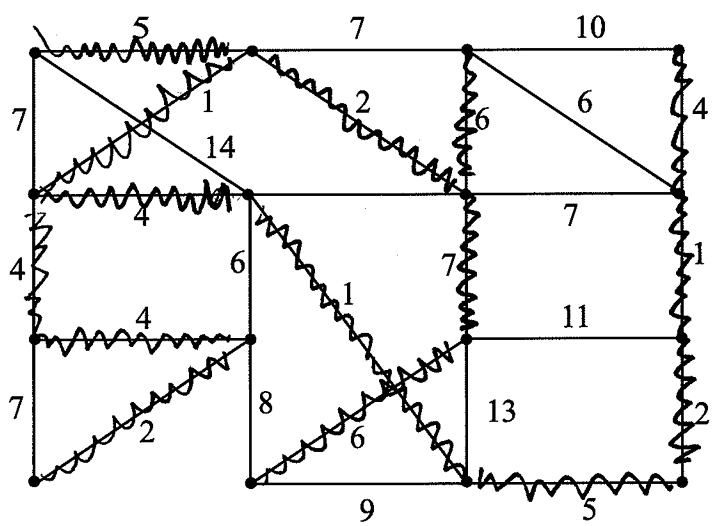
$\rightsquigarrow \chi = 4$.

9. Find the distance between A and Z. Shade in all shortest paths from A to Z.



distance: 16

Shade in a minimal spanning tree for the following graph. What is its weight?



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10. Prove one of the following statements. Circle the statement you are proving.

1. Pigeonhole principle: If m objects are in n boxes, some box has at least $\lceil m/n \rceil$ objects.
2. The cube root of an irrational number is irrational.
3. The square root of 2 is irrational.
4. There are infinitely many prime numbers.
5. There is no bijection between the natural numbers and the real numbers.

Proof by contradiction.

Assume $\sqrt{2}$ is rational, i.e. $\sqrt{2} = p/q$ $p, q \in \mathbb{Z}, q \neq 0$.

$\rightarrow$ $2q^2 = p^2$

LHS has odd # of 2's in prime factorization.

RHS has even #

Contradiction. $\boxed{///}$