# INSOLVABILITY OF THE QUINTIC

References: The Topological proof of the Abel-Ruffini Theorem, Żołądek
Mathematical Omnibus, Fuchs & Tabachnikov (Lecture 5)

Theorem (Abel 1828) There is an integer polynomial of degree 5 whose roots cannot be obtained from the coefficients using only the operations $+, -, \times, \div, \sqrt[k]{\ }$.

Bold idea: prove it by considering the space of all polynomials.

In other words, the polynomial in the theorem is not solvable in radicals.

What is a radical formula?

deg 2: $x^2 + px + q = 0$

$\rightsquigarrow x = -p \pm \sqrt{p^2 - 4q} / 2$

or: $x_1^2 = p^2 - 4q$,

$x_2 = -\frac{1}{2}p + \frac{1}{2}x_1$ ← not unique.

deg 3: $x^3 + px + q$ (special case)

$\rightsquigarrow x_1^2 = p^3/27 + q^2/4$

$x_2^3 = -q/2 + x_1$

$x_4 = x_2 + x_3$

Thm. $x^5 - x - a = 0$ is not solvable in radicals.

# Riemann Surfaces

To understand the roots of $x^5 - x - y = 0$ as $y$ varies we consider all the roots at once:

$$X = \{(x,y) \in \mathbb{C}^2 : x^5 - x = y\}$$

This is just the graph of $y = x^5 - x$. It is called the Riemann surface for $y = x^5 - x$ since it is a surface with a complex structure.

## Simpler examples

$$Z = \{(x,y) : x^2 = y\}$$

The projection $(x,y) \mapsto y$ is a covering map away from $y = 0$, where roots collide. $0$ is a "bad point"

The projection $(x,y) \mapsto x$ shows $Z$ is a surface $\cong \mathbb{C} \setminus \{0\}$.

$$Y = \{(x,y) : x^2 = y^3 - y\}$$

This is a torus minus four points!

Basic idea: $Y$ locally looks like $Z$...

What are the bad pts of $X$? For which $y$ does $x^5 - x = y$ have repeated roots?

algebra calculation $\rightsquigarrow$ $y = \pm \frac{4}{5\sqrt[4]{5}}$ or $\pm \frac{4i}{5\sqrt[4]{5}}$

As we vary $y$ in $\mathbb{C} \setminus \{\text{bad pts}\}$ there are $5$ $x$-values for each $y$

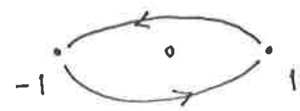
$\rightsquigarrow$ covering space of degree 5.

# MONODROMY

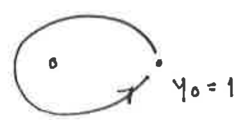
There is a map

$$\pi_1(\mathbb{C} \setminus \text{bad pts}, y_0) \longrightarrow \text{Permutations of roots at } y_0 \leq S_n$$

example. $Z = \{(x,y) : x^2 = y\}$



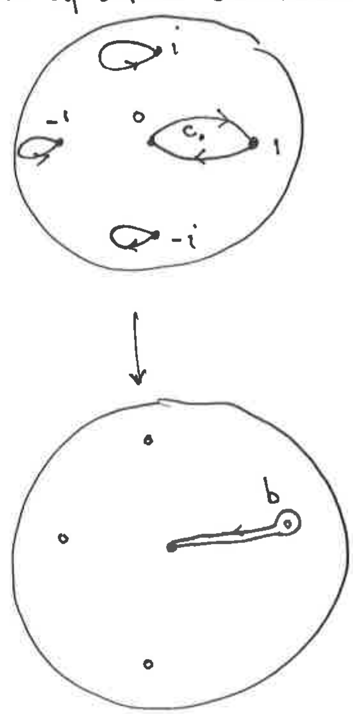

This nontrivial monodromy is exactly why we can't solve quadratics without radicals.




Back to our main example: $X = \{(x,y) : x^5 - x = y\}$

Claim: the monodromy surjects onto $S_5$, i.e. by choosing the right loops, can get any permutation of the roots.

Prove by direct calculation: roots at $0$ are $0, \pm 1, \pm i$.




By symmetry, can swap $0$ with any other root.

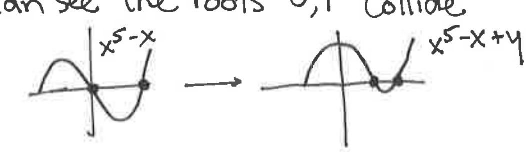
see Fuchs-Tabachnikov

Idea of calculation:

root of $c + \varepsilon \longleftrightarrow y$ of $b - 10c^3\varepsilon^2$

where $c$ = the multiple root above the bad pt over $b = 4/5\sqrt[4]{5}$.

In $\mathbb{R}$-picture, can see the roots $0, 1$ collide as $y$ increases:



# Riemann Surfaces From Radicals

The game now is to show that if the Riemann surface for such an "algebraic function" is built out of radicals, then the monodromy cannot surject onto $S_5$.

A radical formula as above gives a sequence of covers

$$X_N \longrightarrow X_{N-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \mathbb{C} \setminus \{\text{all bad pts}\}$$

Say $X_{i+1}$ obtained from $X_i$ by taking a root: $X_{i+1}^k = X_i$
We claim that the monodromy group (= deck group)
of $X_{i+1} \longrightarrow \mathbb{C} \setminus \{\text{bad pts}\}$ differs from that of
$X_i \longrightarrow \mathbb{C} \setminus \{\text{bad pts}\}$ by $\mathbb{Z}/k\mathbb{Z}$
(definitely true for $i=0$!).

So the monodromy group for $X_N \longrightarrow \mathbb{C} \setminus \{\text{bad pts}\}$ is built out of abelian groups (specifically, cyclic groups).*
Such a group is called <u>solvable</u>.

Precisely, $G$ is solvable if there is a sequence of (normal) subgroups

$$G = G_N \geqslant \cdots \geqslant G_1 \geqslant 1$$

s.t. each $G_{i+1} / G_i$ is abelian.

<u>Thm.</u> $S_5$ is not solvable.

* Need to also think about the other operations besides $\sqrt[k]{\ }$, namely $+, -, \times, \div$. Read the papers or figure it out!