Math 4803
Homework 1 Solution

A1. Show that internal and external presentations of a group G are equivalent. More precisely, show that if we take an internal presentation for G and regard it as an external presentation of a group H, then H is isomorphic to G by the natural isomorphism. And conversely, any external presentation of G naturally gives an internal presentation of G, once we identify the generators of the presentation with their images in G.

We begin with the first direction. Suppose G has internal presentation $\langle S \mid R\rangle$, where S is a generating set and R is a set of defining relations. This means that if two words in $S \cup S^{-1}$ are equal then they differ by a sequence of substitutions using elements of R.

As a running example, take $G = \mathbb{Z}^2$ with the presentation $\langle a, b \mid ab = ba\rangle$. In this presentation, a and b are taken to *be* $(1, 0)$ and $(0, 1)$, respectively. Using the relation we can transform aba to baa, etc.

Let H be the group given by considering the internal presentation as an external one. In our running example, this means that $H = F(\tilde{S})/\left\langle\left\langle \tilde{R}\right\rangle\right\rangle$, where $\tilde{S}$ is another copy of S, $\tilde{R}$ is the corresponding set of relators in $F(\tilde{S})$, and $\left\langle\left\langle \tilde{R}\right\rangle\right\rangle$ means the normal closure of $\tilde{R}$ in $F(\tilde{S})$, the free group on $\tilde{S}$. In our running example, $\tilde{S} = \{\tilde{a}, \tilde{b}\}$, $\tilde{R} = \{\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\}$, and $H = F(\tilde{a}, \tilde{b})/\left\langle\left\langle \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1}\right\rangle\right\rangle$.

We would like to show that H is isomorphic to G. There is a natural homomorphism from $\phi : H \to G$ sending generators in $\tilde{S}$ to the corresponding generators in S. In our example, the homomorphism sends $\tilde{a}$ and $\tilde{b}$ to a and b. (By the universal property of free groups, you can define a homomorphism from a free group to any other group by declaring where the generators go; it is a good exercise to prove this.)

By the first isomorphism theorem, it suffices to show that the kernel of ϕ is $\left\langle\left\langle \tilde{R}\right\rangle\right\rangle$. It is straightforward to see that $\tilde{R}$ is contained in the kernel. Since the kernel is normal it follows that $\left\langle\left\langle \tilde{R}\right\rangle\right\rangle$ is contained in the kernel, as this is the smallest normal subgroup containing $\tilde{R}$.

The key step is showing that the kernel of ϕ is contained in $\left\langle\left\langle \tilde{R}\right\rangle\right\rangle$. Suppose that $\phi(\tilde{w})$ is trivial. We need to show that $\tilde{w}$ is a product of conjugates of elements of $\tilde{R}$ (this is another characterization of $\left\langle\left\langle \tilde{R}\right\rangle\right\rangle$). This is a consequence of the following statement: if two words v and w in S differ by a relation in R, then the corresponding elements of $F(\tilde{S})$ differ by a conjugate of an element of $\tilde{R} \cup \tilde{R}^{-1}$, meaning that $\tilde{v} = \tilde{w}\tilde{r}$ where $\tilde{r}$ is a conjugate of an element of $\tilde{R} \cup \tilde{R}^{-1}$.

As an example, consider in our running example the elements aba and baa. These differ by the relation $ab = ba$. The corresponding elements of $F(\tilde{S})$ are $\tilde{a}\tilde{b}\tilde{a}$ and $\tilde{a}\tilde{a}\tilde{b}$. We see that

$$\tilde{a}\tilde{b}\tilde{a} = (\tilde{a}\tilde{a}\tilde{b})(\tilde{b}^{-1}\tilde{a}^{-1}\tilde{b}\tilde{a}) = (\tilde{a}\tilde{a}\tilde{b})\left((\tilde{b}\tilde{a})^{-1}(\tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1})^{-1}(\tilde{b}\tilde{a})\right),$$

as desired.

Suppose as above that $v = w_s v_m w_e$ and $w = w_s w_m w_e$ (start, middle, end), and that $v_m = w_m$ is a relation in R (we allow any of the three parts of each word to be empty). In other words, v and w are words in S that differ by a relation. Let $\tilde{r}$ be the corresponding relator (or inverse of a relator!) $\tilde{w}_m^{-1}\tilde{v}_m$ in $\tilde{R}$. Then we have

$$\tilde{w}(\tilde{w}_e^{-1}\tilde{w}_m^{-1}\tilde{v}_m\tilde{w}_e) = (\tilde{w}_s\tilde{w}_m\tilde{w}_e)(\tilde{w}_e^{-1}\tilde{w}_m^{-1}\tilde{v}_m\tilde{\tilde{}}w_m) = (\tilde{w}_s\tilde{v}_m\tilde{w}_e) = \tilde{v},$$

as desired (here, putting a tilde on a word means the natural thing). (Make sure you see why this finishes the proof!)

For the other direction we start with an external presentation $\langle \tilde{S} \mid \tilde{R} \rangle$. Let S denote the image of $\tilde{S}$ in the quotient $G = F(S)/\left\langle\left\langle \tilde{R} \right\rangle\right\rangle$. Since quotient maps are surjective, we have that S is a generating set for G. Let R be the relations in G which say that the words in S corresponding to $\tilde{R}$ are equal to the identity (we could also use other relations, for example replacing $aba^{-1}b^{-1}$ with $ab = ba$, etc.).

We need to check that R is a defining set of relations for G. That is, if v and w are words in S that are equal, then they differ by a finite sequence of relations, as in the proof above. If $v = w$ in G, it follows from the definition of G as a quotient that $\tilde{v} = \tilde{w}\tilde{p}$, where $\tilde{p}$ is a product of conjugates of elements of $\tilde{R} \cup \tilde{R}^{-1}$.

Let us consider the case where $\tilde{p}$ is a single conjugate of an element of $\tilde{R}$, that is, $\tilde{p} = \tilde{a}\tilde{r}\tilde{a}^{-1}$. We then have

$$\tilde{v} = \tilde{w}\tilde{p} = \tilde{w}\tilde{a}\tilde{r}\tilde{a}^{-1}.$$

The corresponding equalities in G are

$$v = wp = wara^{-1}.$$

In other words, what is happening in G is that we replace w with waa^{-1}, which is allowed because aa^{-1} is an automatic relation. And then we insert r between the a and the a^{-1}, which is allowed because $r = \mathrm{id}$ is a relation. So this is an application of one free (un)cancellation and one relation.

If we have a more complicated product of conjugates of elements of $\tilde{R} \cup \tilde{R}^{-1}$, it works the same; it's just a repeated application of relations, as above. This completes the proof.