Mathematics 4803

Midterm
Prof. Margalit
March 2021

This is a take home exam, assigned on March 4, 2021 and due at 3:30 on March 11, 2021 on Gradescope. You must work individually on this assignment. You may use your notes, the textbook, recordings from class, Piazza, and the Office Hours book. You should not discuss the exam with classmates or use GroupMe to discuss the exam. You may not search the internet for hints.

1. Suppose that a group G acts on a set X, that $n \geq 2$, that $g_1, \ldots, g_n$ are elements of G, and that $X_1, \ldots, X_n$ are nonempty disjoint subsets of X with $g_i^k(X \setminus X_i) \subseteq X_i$ for all nonzero k and all i. Suppose further that there is a function $h : X \to \mathbb{R}_{\geq 0}$ with the property that $h(g_i^k(x)) > h(x)$ for all i, all $x \in X \setminus X_i$, and all nonzero k. Prove that the subgroup H of G generated by $g_1, \ldots, g_n$ is isomorphic to the free group F_n and moreover that if an element of H is not conjugate to one of the g_i then the action of the corresponding cyclic subgroup on X has no periodic points. (Each cyclic subgroup of H is of the form $\{h^i \mid i \in \mathbb{Z}\}$, and a point $x \in X$ is periodic if $h^i \cdot x = h^j \cdot x$ for some $i \neq j$.)

Use the first part of the problem to prove that the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ is isomorphic to F_2 and that cyclic subgroups not generated by conjugates of the given matrices have no periodic points in $\mathbb{Z}^2 \setminus \{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)\}$.

2. Suppose that G acts on a graph Γ and that G is generated by $S \subseteq G$. Suppose that v is some vertex of Γ, that each component of Γ contains a point of the G–orbit of v, and that for each $s \in S$ there is a path in Γ from v to $s \cdot v$. Prove that Γ is connected.

Use the first part of the problem and the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ to show that the Farey graph is connected.

3. Consider the action of $\mathrm{SL}_2(\mathbb{Z})[2]$ on the Farey graph. Show that there is a fundamental domain consisting of one triangle (in class, we found a fundamental domain for the Farey complex, which was two 2D triangles).

Use the first part of the problem to find a generating set for $\mathrm{SL}_2(\mathbb{Z})[2]$.

4. Tile the Euclidean plane by squares. Let W be the group generated by reflections in the four lines containing the sides of one square. Draw the Cayley graph of W and show that $W \cong D_\infty \times D_\infty$.

5. Let $F_2 = \langle a, b \rangle$. Suppose x and y are elements of F_2 given by $x = wa^kw^{-1}$ and vb^jv^{-1} for some $v, w \in F_2$ and $j, k \neq 0$. Use the tree for $\mathbb{Z} * \mathbb{Z}$ to show that $\langle x, y \rangle \cong F_2$.