ANNOUNCEMENTS APR 6

- Cameras on
- HW due Thu (your choice of 2)
- Draft due Fri
- Office Hours <u>moved</u> Wed 2-3, Tue 11-12, appt
- Makeup work

<u>Today</u>

Quasi-isometries

GEOMETRY VS. ALGEBRA

<u>Thm.</u> $G \underset{QI}{\cong} \mathbb{Z} \Longrightarrow G$ has finite index subgp $H \cong \mathbb{Z}$.

geometry

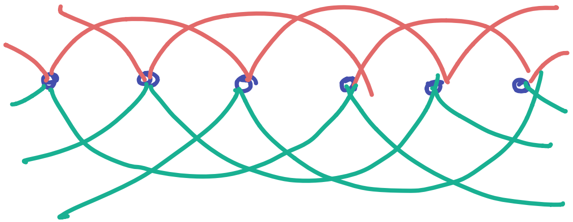
algebra

Two Cayley graphs for $\mathbb{Z}$

$S=\{1\}$

$d(-2,5)=7$

$S=\{2,3\}$



$d(-2,5)=3$

Looks like $\mathbb{R}$ from far away.

Will show: it is QI to $\mathbb{R}$.

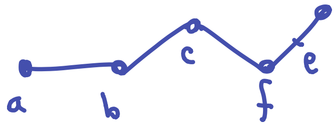
Metric space

(X, d_X) (Y, d_Y) metric spaces.

meaning $d_X(x_1, x_2)$ is distance.

- $d(x,y) = 0 \Leftrightarrow x = y$
- $d(x,y) + d(y,z) \geq d(x,z)$
- $d(x,y) = d(y,x)$

examples: graphs



$d(a,c) = 2$

$d(a,e) = 3\frac{1}{2}$

groups word metric/ distance in Cayley graph.

Isometries

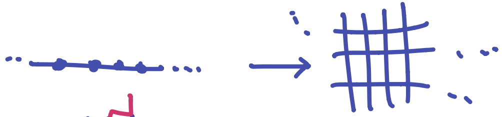
Equivalence of metric spaces

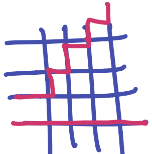
$f: X \to Y$ is an isometric embedding if it doesn't change distances:

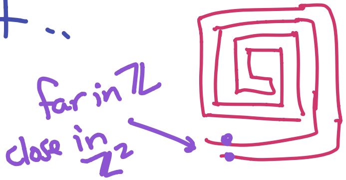
$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

($\Rightarrow$ injective)

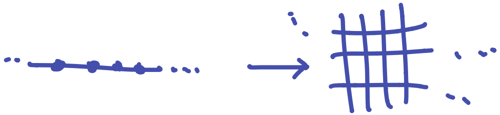
If f is also surjective, say f is an <u>isometry</u>.

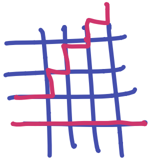
<u>Examples</u>. f: 

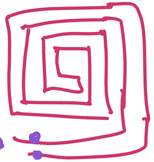
~~An~~ Two isom. emb. 

Not: 

Examples

① $f:$ 

~~An~~ Two isom. emb. 

Not: far in $\mathbb{Z}$ close in $\mathbb{Z}^2$ 

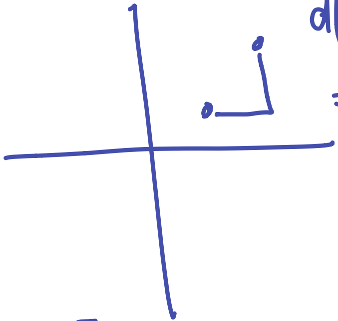
or any non-inj.

② $\mathbb{Z}^2 \xrightarrow{\text{inclusion}} \mathbb{R}^2$

Is this an isom. emb?

No with std metric on $\mathbb{R}^2$

Yes with "taxicab metric" on $\mathbb{R}^2$



$d((x_1,y_1),(x_2,y_2)) = |x_1-x_2| + |y_1-y_2|$

③ identity $\mathbb{Z} \to \mathbb{Z}$
$n \mapsto n$

where first $\mathbb{Z}$ has $\{1\}$ metric
second $\mathbb{Z}$ has $\{2,3\}$ metric

Not an isom. emb.

Bi-Lipschitz equivalence

$f : X \to Y$ is a bi-Lipschitz embedding if $\exists\ K \geq 1$ s.t.

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K\, d_X(x_1, x_2)$$

(K indep. of x_1, x_2).

If f also surj. then f is $\underbrace{\text{bi-Lip equiv.}}_{\text{equiv. rel.}}$

Thm. G = group

S, S' two finite gen sets

$\mathrm{id} : (G, d_S) \longrightarrow (G, d_{S'})$

is a bi-Lip eq

Example

① $(\mathbb{Z}^2, \text{std}) \longrightarrow (\mathbb{R}^2, \text{std})$

$K = \sqrt{2}$ bi-Lip emb

Thm. G = group

S, S' two finite gen sets

$id : (G, d_S) \to (G, d_{S'})$

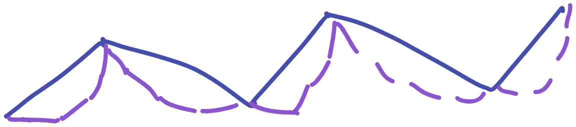
is a bi-Lip eq

vertex set of Cayley graph.
If you include the edges, lose bi-Lip equiv.

Pf idea What is K?

$$K = \max \{ d_{S'}(id, s) : s \in S \} \cup \{ d_S(id, s) : s \in S' \}$$

Use triangle inequality.



Quasi-isometries

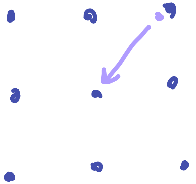
$f : X \to Y$ is a q.i. emb if $\exists\ K \geq 1, C \geq 0$ s.t.

$$\frac{1}{K} d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq K\, d(x_1, x_2) + C$$

If $\exists\ D \geq 0$ s.t. each pt of Y is within D of $f(X)$ then f is a quasi-isometry.

Examples. ① $(\mathbb{Z}^2, \text{std}) \longrightarrow (\mathbb{R}^2, \text{std})$

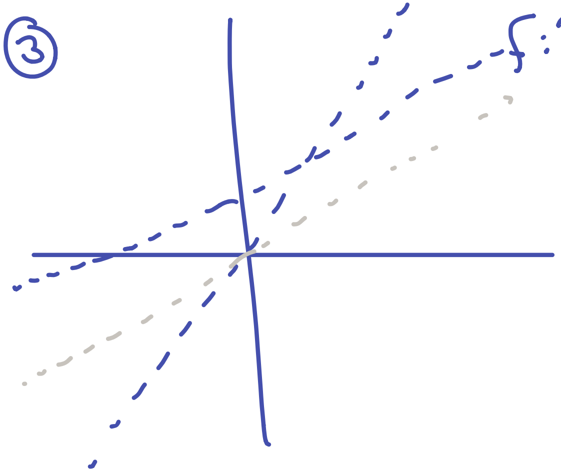
$K = \sqrt{2}$ $\quad C = 0$ $\quad D = \sqrt{2}/2$



② $(\mathbb{R}^2, \text{std}) \longrightarrow (\mathbb{Z}^2, \text{std})$ $\quad K = \sqrt{2}$ $\quad D = 0.$

$(x, y) \longmapsto (\lfloor x \rfloor, \lfloor y \rfloor)$ $\quad C = \sqrt{2}$

3

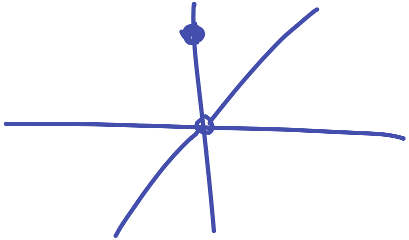


$f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases}5x & x\in\mathbb{Q}\\ 3x+1 & x\notin\mathbb{Q}\end{cases}$$

$K=5$ $C=1$ $D=1.$ (or 0)

4 $$f(x)=\begin{cases}5x & x\neq 0\\ 7 & x=0\end{cases}$$

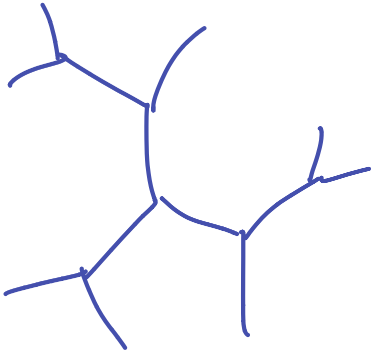


$K=5$

$C=7$

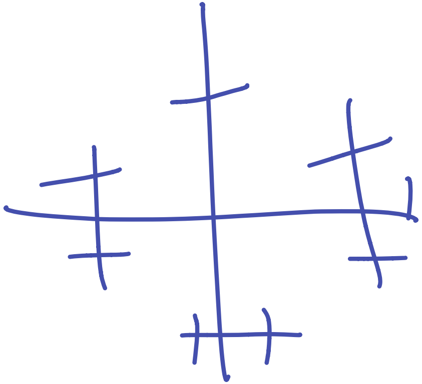
$D=1$

Poll.



T_3

$\overset{QI}{\simeq}$



T_4

?

Yes. Ponzi scheme.

QI's are violent, but still have thm: $G \underset{QI}{\simeq} \mathbb{Z}$ then G has $\underset{\mathbb{Z}}{\underset{\cong}{H}} \overset{fi}{\leq} G$