Announcements Apr 8

- Cameras on
- HW due Thu - I forgot again!
- First draft due Fri - share on Teams & Reviews
- Office Hours Tue 11, appt. (more soon!)
- Makeup points

Today

- Milnor-Schwarz Lemma
- $G \underset{QI}{\simeq} \mathbb{Z} \Rightarrow G$ is virtually $\mathbb{Z}$

Quasi-isometries

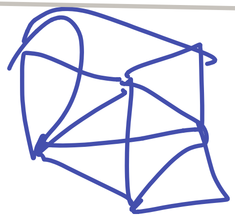
(X, d_X), (Y, d_Y) metric spaces

$f : X \to Y$ is a quasi-isom. if

$$\frac{1}{K} d_X(x_1, x_2) - C \le d_Y(f(x_1), f(x_2)) \le K d_X(x_1, x_2) + C$$

and there is a D so all pts of Y are within distance D of $f(X)$

F = finite gp



$\underset{QI}{\simeq}$ $\Gamma = \bullet$ with $K = 1$, $C = \mathrm{diam}(F)$, $D = 0$

MILNOR-SCHWARZ LEMMA (Graph version)

(Fund Lemma of GGT)

<u>Thm</u>. $G \curvearrowright \Gamma$ = graph

geometric action: $\left\{ \begin{array}{l} \text{finite fund dom} \\ \text{(or } \Gamma/G \text{ finite)} \\ \text{Action is prop. disc.} \end{array} \right.$

Then: G is fin. gen.

& $G \underset{QI}{\simeq} \Gamma$.

$G \curvearrowright \Gamma$ is properly discontinuous if $\forall$ $K \subseteq \Gamma$ finite subgraph

$\#\{g \in G : g \cdot K \cap K \neq \emptyset\} < \infty$.

P.D. and finite f.d. Needed for Thm because... might $G \curvearrowright \Gamma$ trivially.

Example. ⓪ F = finite gp $\Gamma = \bullet \Rightarrow F \underset{QI}{\simeq} \bullet$

with $K =$

$C =$

So: All finite gps are QI to trivial gp.

Milnor-Schwarz Lemma

(Fund Lemma of GGT)

Thm. $G \circlearrowright \Gamma$ = graph

finite fund dom

(or Γ/G finite)

Action is prop. disc.

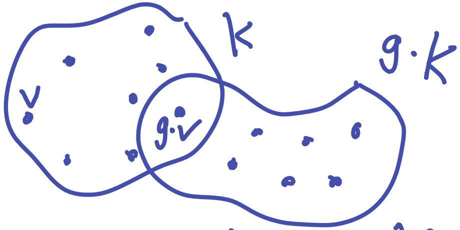
Then: G is fin. gen.

& $G \underset{QI}{\simeq} \Gamma$.

Examples. ① $\mathbb{Z} \circlearrowright$ 

② $SL_2(\mathbb{Z}) \circlearrowright$ Farey tree.

Prop. disc: a $\overset{\text{deg } 2}{\text{vertex}}$ of Γ is a basis for $\mathbb{Z}^2$



means

g took a basis in K to another basis in K.

$\Longrightarrow SL_2\mathbb{Z} \underset{QI}{\simeq} T_3 \underset{QI}{\simeq} T_4 \underset{QI}{\simeq} F_2 \underset{QI}{\simeq} F_k \quad k \geq 2$

Applications

① $H \leq G$ finite index.

$H \curvearrowright \Gamma$ = Cayley graph for G.

with finite fund. dom

& prop. disc (b/c G acts p.d.)

$\Longrightarrow H \underset{QI}{\simeq} G$.

② $N \trianglelefteq G$ N finite.

$G \curvearrowright \Gamma$ = Cayley graph for G/N.

with p.d, & finite fund dom.

b/c N is finite.

$\Longrightarrow G/N \underset{QI}{\simeq} G$

We say two groups differ by finite gps if can get from one to other by taking finite index subgps & quotients by finite groups.

[③ $(G,S) \underset{QI}{\simeq} (G,S')$]

Gromov's Program

Which fin. gen. gps are quasi-isometric?

We saw: G, H differ by finite groups $\Longrightarrow G \underset{QI}{\simeq} H$

We say G is quasi-isometrically rigid if $G \underset{QI}{\simeq} H \Longrightarrow G, H$ differ by finite gps.

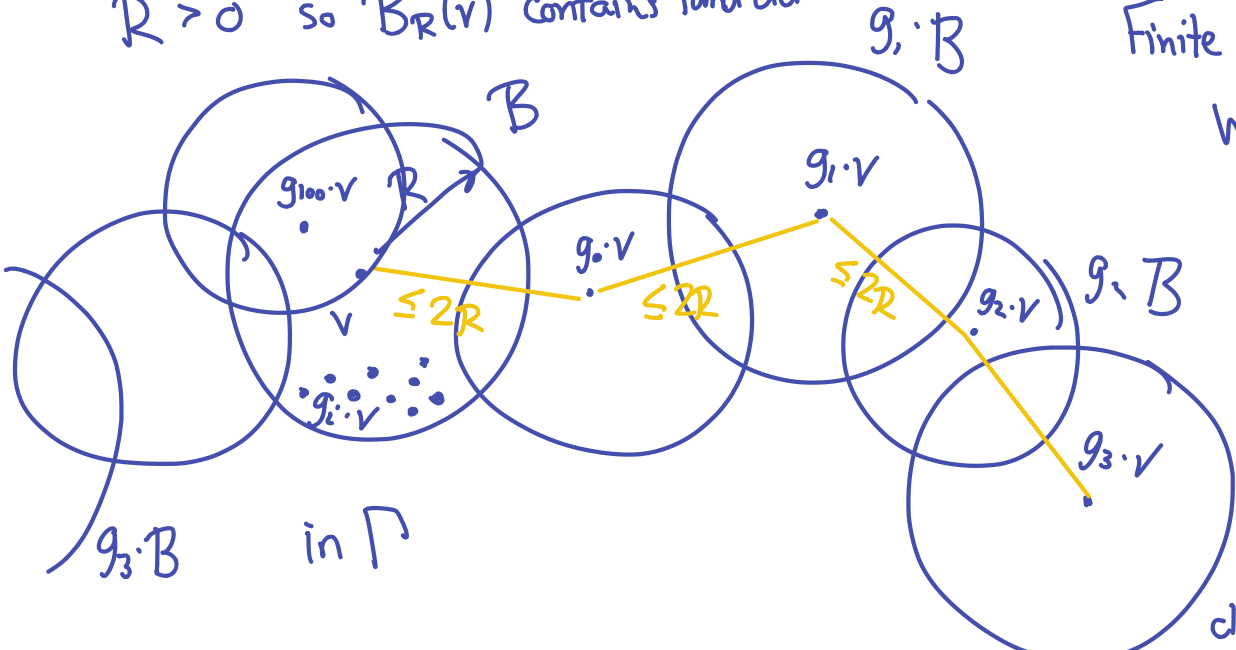
Examples

0. Trivial gp.
1. $\mathbb{Z}^n$ (we'll prove $n=1$ case soon)
2. Braid groups. & Mapping class groups.
3. Free groups.

Idea of Milnor-Schwarz

Choose vertex v.

$R > 0$ so $B_R(v)$ contains fund dom.



Get the finite gen set as usual

$S = \{ g : g \cdot B \cap B \neq \emptyset \}$ ✓

Finite by prop disc.

What are K, C, D?

Distances in Γ not much longer than in G by defn of R.

Opposite direction: If have g_i with $|g_i| \to \infty$ & $g_i \cdot v$ close to v, violate P.D.

Thm. G = fin. gen. gp.

$G \underset{QI}{\simeq} \mathbb{Z}$

$\Rightarrow \exists\, H \underset{fi}{\leq} G,\ H \cong \mathbb{Z}$

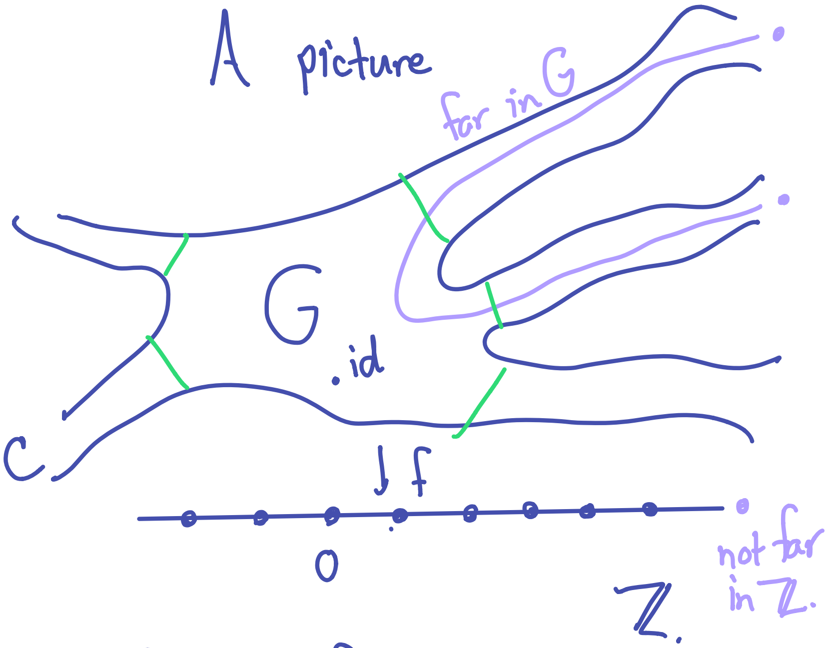
Pf. Let $f: G \to \mathbb{Z}$.

$\frac{1}{K} d(x,y) - C \leq |f(x) - f(y)| \leq K\, d(x,y) + C$

WLOG $f(\mathrm{id}) = 0$.

(change C if needed).

A picture



First Goal: 2 fingers, not 5

one left

one right.

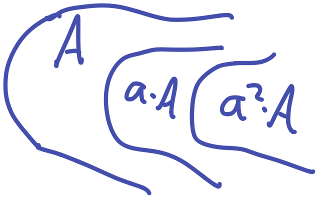
Step 1. G has ∞ order elt a.

Step 2. $|G/\langle a\rangle| < \infty$

For step 1, find $A \subseteq G$, $a \in G$

s.t. $a \cdot A \subsetneq A$

$\Longrightarrow |a| = \infty$.



How to find A?

Let $L \gg K, C$ $\quad (L = K + C)$

$B = f^{-1}([-L/2, L/2])$