Announcements Apr 13

- Cameras on
- Last HW due Thu
- Peer evaluations due Fri
- Presentations next week ~20
- Final draft due Apr 27 3:30.
- Makeup problems
- CIOS

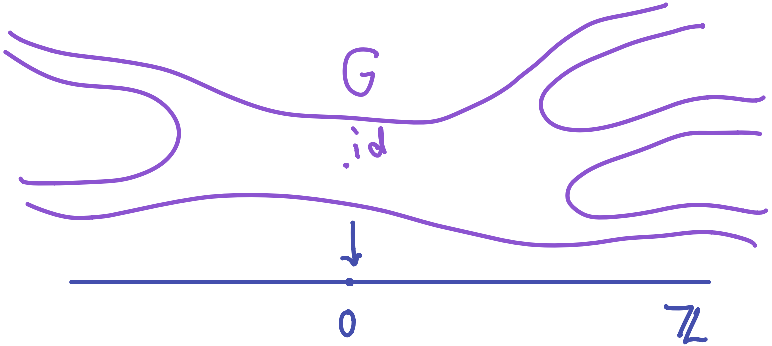
Today

- $G \underset{QI}{\cong} \mathbb{Z} \Rightarrow G \overset{\cong}{\underset{v}{\geq}} \mathbb{Z}$
- Ends of groups:
  Freudenthal-Hopf Thm

Thm. G = fin gen. gp

$$G \underset{QI}{\simeq} \mathbb{Z}$$

Then G has finite index subgp $H \cong \mathbb{Z}$.



Pf. Let $f: G \longrightarrow \mathbb{Z}$ qi.

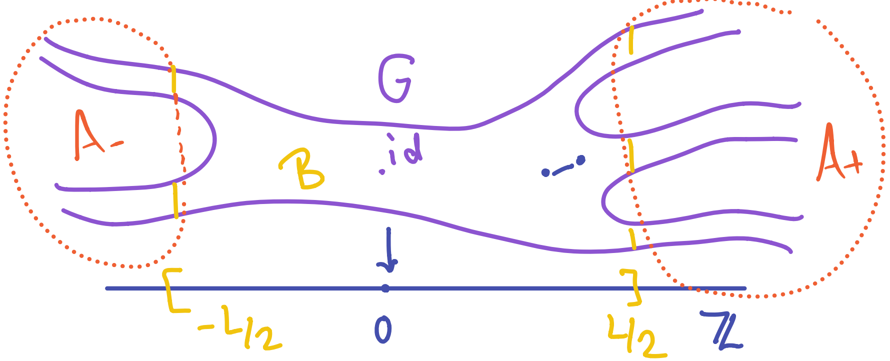
$$\tfrac{1}{K}\, d(x,y) - C \leq |f(x) - f(y)| \leq K d(x,y) + C$$

Also: D...

WLOG $f(\mathrm{id}) = 0$.

Step 1. G has ∞ order elt a

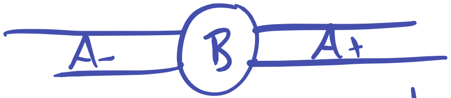
Step 2. $\langle a \rangle$ has finite index in G.



WLOG $G \setminus B$ has only unbounded pieces (if not, add any bounded pieces to B)

Let $A_+ = f^{-1}(L/2, \infty) \setminus B$
$A_- = f^{-1}(-\infty, -L/2) \setminus B$

Want this pic:



or: A_+, A_- each connected.
also, separate from eachother.

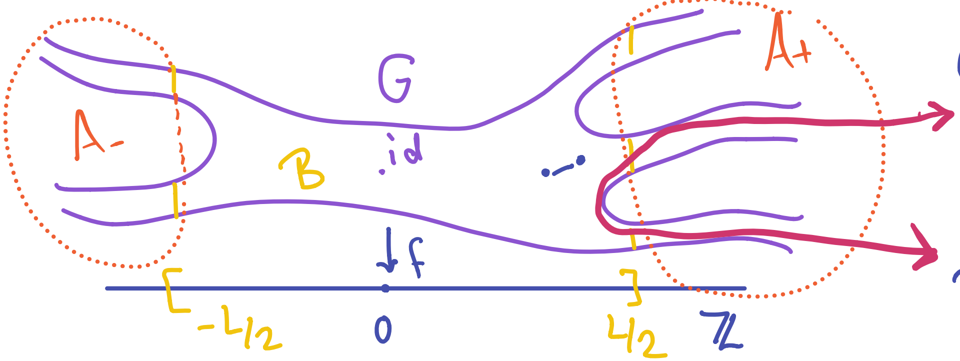
Step 1. G has ∞ order elt a

Suffices to find $A \subseteq G$, $a \in G$
$a \cdot A \subsetneq A$ (ping pong)

Let $L = K + C$ (e = edge in G $\Rightarrow f(e)$ has length $\leq L$)

$B = f^{-1}([-L/2, L/2])$

Note: if g, h connected in G, can't lie on opp. sides of B.



Claim 1 $G \setminus B$ has ≥ 2 pieces

i.e. A_+, A_- not connected to each other.

Pf. The above note.

Claim 2. $G \setminus B$ has ≤ 2 pieces.

Pf. Otherwise find arbitrarily far pt of G mapping to same pt of $\mathbb{Z}$. $\square$

Let $L = K + C$ (e = edge in G $\Rightarrow$ $f(e)$ has length $\leq L$)

$$B = f^{-1}([-L/2, L/2])$$

Note: if g, h connected in G, can't lie on opp. sides of B.

Let $A_+ = f^{-1}(L/2, \infty) \setminus B$

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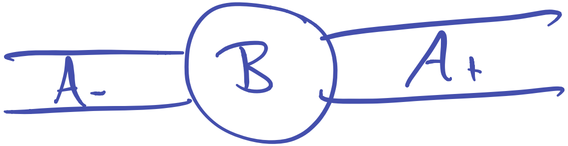
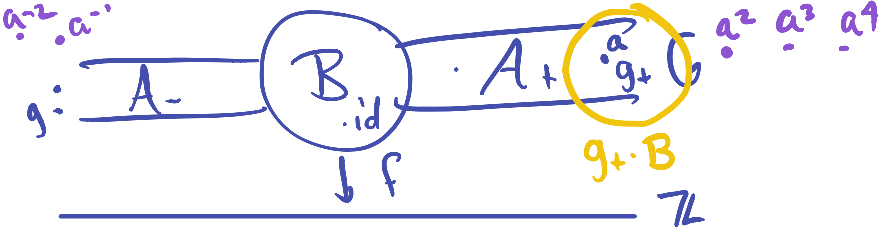
Now Have:





Let $g,h \in G$ s.t. $g_+ \in A_+$, $g_- \in A_-$

$$d(id, g_\pm) > 2\,\mathrm{diam}(B)$$

Claim 3. For some $a \in \{g_+, g_-, g_+g_-\}$

$$a \cdot A_\pm \subsetneq A_\pm$$

Pf. Case 1. $g_+ \cdot A_+ \subseteq A_+$ ✓

Case 2. $g_- \cdot A_- \subseteq A_-$ ✓

Can we argue these cases are the same?

Case 3. Neither true. Think D_∞! $\square$

Step 2. $\langle a \rangle$ has finite index in G.

Claim 1. $d(id, a^n) \to \infty$ as $n \to \infty$

Claim 2. $\exists D$ s.t. D nbd of $\langle a \rangle$ in G is G.

Claim 3. $|G/\langle a \rangle| < \infty$.

Pf of Claim 1: By Step 1, a^n all distinct, but G locally finite (where we use G fin gen.).

Pf of Claim 2. Claim 1 $\Rightarrow$

$$d(a^m, a^n) \to \infty \quad |m-n| \to \infty$$

$$\Rightarrow f(a^i) \to \infty \quad f(a^{-i}) \to -\infty$$

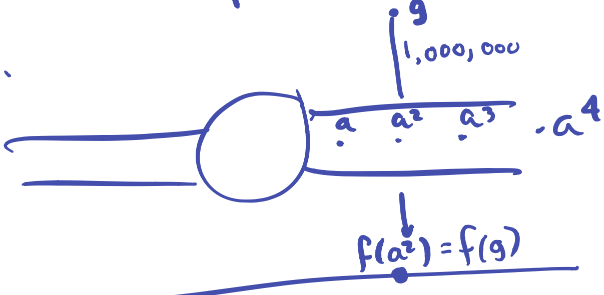
Claim 2. $\exists D$ s.t. D nbd of $\langle a \rangle$ in G is G.

Pf of Claim 2. Claim 1 $\Rightarrow$

$d(a^m, a^n) \to \infty \quad |m-n| \to \infty$

$\Rightarrow f(a^i) \to \infty \quad f(a^{-i}) \to -\infty$

If there were pts in G arbit. far from $\langle a \rangle$ then arb far pts in G would map to "same" pt in $\mathbb{Z}$.



Claim 3. $|G/\langle a \rangle| < \infty$.

Let Γ = Cayley graph for G

$\Gamma/\langle a \rangle$ has one vertex for all a^n

& locally finite,

& finite diam by Claim 2

$\Rightarrow \Gamma/\langle a \rangle$ finite

But vertices of $\Gamma/\langle a \rangle$ are the cosets of $\langle a \rangle$ in G. $\square$

Ends of Groups

Freudenthal-Hopf Thm

G = fin gen gp

$\Longrightarrow$ G has $0, 1, 2,$ or (∞ many) ends

Some defns:

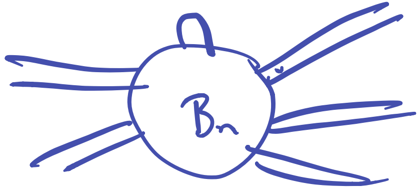
Γ = connected graph, locally finite.

v = base vertex.

B_n = ball of radius n around v.

$\|\Gamma \setminus B_n\|$ = # unbounded pieces of $\Gamma \setminus B_n$.

Γ

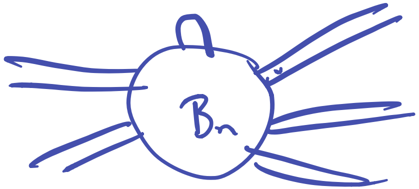


$\|\Gamma / B_n\| = 5$

$$e(\Gamma) = \lim_{n \to \infty} \|\Gamma \setminus B_n\|.$$

$\|\Gamma \setminus B_n\|$ = # unbounded pieces of $\Gamma \setminus B_n$.

Γ

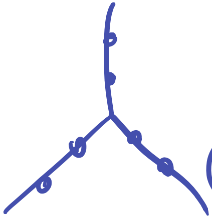


$\|\Gamma \setminus B_n\| = 5$

$$e(\Gamma) = \lim_{n \to \infty} \|\Gamma \setminus B_n\|.$$

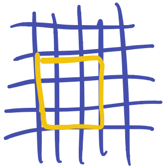
Examples ① Γ finite.

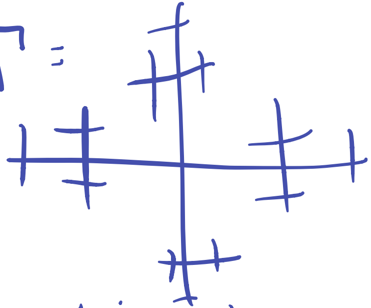
$\Longrightarrow e(\Gamma) = 0$



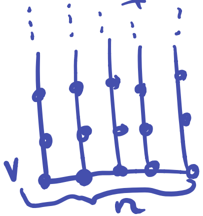
② $\Gamma =$ 

$e(\Gamma) = 2$

③ $\Gamma =$  $e(\Gamma) = 1$

④ $\Gamma =$ 

$e(\Gamma) = \infty$

⑤ $\Gamma =$ 

$e(\Gamma) = n$.