Announcements April 13

- Cameras on
- Last HW due Thu
- Peer evaluations due Fri
- Presentations next week ~20
- Final draft due Apr 27 3:30.
- Makeup problems
- CIOS

Today
- \( G_{\approx} \mathbb{Z} \Rightarrow G \cong \mathbb{Z} \)
- Ends of groups: Freudenthal-Hopf Thm
Thm. $G = \text{fin gen. gp}$

$G \cong \mathbb{Z}$

Then $G$ has finite index

$\text{subgp } H \cong \mathbb{Z}$.

Pf. Let $f: G \to \mathbb{Z}$ be $g_i$.

\[
\frac{1}{k} d(x,y) - C \leq |f(x) - f(y)| \leq k d(x,y) + C
\]

Also: D...

WLOG $f(\text{id}) = 0$.

Step 1. $G$ has $\infty$ order elt $a$

Step 2. $\langle a \rangle$ has finite index in $G$. 
WLOG $G \setminus B$ has only unbounded pieces (if not, add any bounded pieces to $B$)

Let $A_+ = f^{-1}(L_2, \infty) \setminus B$

Let $A_- = f^{-1}(-\infty, -L_2) \setminus B$

Want this pic:

\[ \begin{array}{c}
A_- & B & A_+ \\
\end{array} \]

or: $A_+$, $A_-$ connected. each also, separate from eachother.

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Step 1. $G$ has $\infty$ order elt $a$

Suffices to find $A \subseteq G$, $a \in G$

$a \cdot A \subseteq A$ (ping pong)

Let $L = K + C$ ($e = \text{edge in } G$

$\Rightarrow f(e) \text{ has length } \leq L$)

$B = f^{-1}([-L_2, L_2])$

Note: if $g, h$ connected in $G$, can't lie on opp. sides of $B$. 
Let $L = K + C$ \quad (e = \text{edge in } G \implies f(e) \text{ has length } \leq L)

$B = f^{-1}([-L/2, L/2])$

Note: if $g, h$ connected in $G$, can’t lie on opposite sides of $B$.

Let $A_+ = f^{-1}([L/2, \infty)) \setminus B$

$A_- = f^{-1}((\infty, -L/2)) \setminus B$

Claim 1: $G \setminus B$ has $\geq 2$ pieces
\quad i.e. $A_+, A_-$ not connected to each other.

\textbf{Pf.} The above note.

Claim 2: $G \setminus B$ has $\leq 2$ pieces.

\textbf{Pf.} Otherwise find arbitrarily far pt of $G$ mapping to same pt of $\mathbb{Z}$. \hfill \Box

Now Have:

\[ \begin{array}{c}
A_+ \\
B \\
A_-
\end{array} \]
Let $g, h \in G$ s.t. $g \in A^+, g_\infty (A)$.

d$(id, g^+ ) > 2 \text{diam}(B)$

Claim 3. For some $a \in \{g^+, g_\infty, g^+g^-\}$

\[ a \cdot A^+ \not\subseteq A^+ \]

Case 1. $g^+ \cdot A^+ \subseteq A^+$

Case 2. $g^- \cdot A^- \subseteq A^-$

Case 3. Neither true. Think $D_\infty$!

Step 2. $\langle a \rangle$ has finite index in $G$.

Claim 1. $d(id, a^n ) \to \infty$ as $n \to \infty$

Claim 2. $\exists D$ s.t. $D$ nbd of $\langle a \rangle$ in $G$ is $G$.

Claim 3. $|G/\langle a \rangle| < \infty$.

Pf of Claim 1: By Step 1, $a^n$ all distinct, but $G$ locally finite (where we use $G$ fin gen.).

Pf of Claim 2. Claim 1 $\Rightarrow$

\[ d(a, a^n) \to \infty \text{ as } n \to \infty \]

$\Rightarrow f(a^+) \to \infty \text{ if } f(a^+) \to -\infty$
Claim 2. \( D \) s.t. \( D \) nbhd of \(<a>\) in \( G \) is \( G \).

**Pf of Claim 2.** Claim 1 \( \Rightarrow \)
\[
d(a^m, a^n) \to \infty \text{ as } |m-n| \to \infty
\]
\[
\Rightarrow f(a^i) \to \infty \text{ as } f(a^{-i}) \to -\infty
\]

If there were pts in \( G \) arbit f.r.
\[
\text{far from } <a> \text{ then arb for}
\text{pts in } G \text{ would map to "same"}
\text{pt in } Z.
\]

Claim 3. \(|G/<a>|< \infty\).

Let \( \Gamma = \text{Cayley graph for } G \)
\[
\Gamma/<a> \text{ has one vertex for all } a
\]
& locally finite, & Finite diam by Claim 2
\[
\Rightarrow \Gamma/<a> \text{ finite}
\]

But vertices of \( \Gamma/<a> \)
\[
\text{are the cosets of } <a> \text{ in } G.
\]

\[\square\]
Ends of Groups

Freudenthal-Hopf Thm

\[ G = \text{fin gen gp} \]

\[ \Rightarrow G \text{ has } 0, 1, 2, \text{ or } (\infty \text{ many}) \text{ ends} \]

Some defs:

\[ \Gamma = \text{connected graph, locally finite} \]

\[ V = \text{base vertex} \]

\[ B_n = \text{ball of radius } n \text{ around } V \]

\[ \| \Gamma \setminus B_n \| = \# \text{ unbounded pieces of } \Gamma \setminus B_n. \]

\[ e(\Gamma) = \lim_{n \to \infty} \| \Gamma \setminus B_n \|. \]
\[ \| \Gamma \setminus B_n \| = \# \text{ unbounded pieces of } \Gamma \setminus B_n. \]

\[ e(\Gamma) = \lim_{n \to \infty} \| \Gamma \setminus B_n \|. \]

**Examples**

1. \( \Gamma \) finite.
   \[ \Rightarrow e(\Gamma) = 0 \]

2. \( \Gamma = \quad \)
   \[ e(\Gamma) = 2 \]

3. \( \Gamma = \quad \)
   \[ e(\Gamma) = 1 \]

4. \( \Gamma = \quad \)
   \[ e(\Gamma) = \infty \]

5. \( \Gamma = \quad \)
   \[ e(\Gamma) = n \]